The impact of statistical learning on violations of the sure-thing principle*

Nicky Nicholls† Aylit Romm‡ Alexander Zimper§

August 16, 2013

Abstract

This paper experimentally tests whether violations of Savage's (1954) subjective expected utility theory decrease if the ambiguity of an uncertain decision situation is reduced through statistical learning. Because our data does not show such a decrease, existing models which formalize ambiguity within an Anscombe-Aumann (1963) framework—thereby reducing to expected utility theory in the absence of ambiguity—are violated. Models are needed that allow for a violation of von Neumann and Morgenstern's (1947) independence axiom whenever uncertain decision situations transform into risky decision situations for which probabilities are known.

Keywords: Choquet Expected Utility Theory; Multiple Priors Expected Utility Theory; Sure Thing Principle; Independence Axiom
JEL Classification Numbers: C91, D81

*We want to thank Justine Burns and Ulrich Schmidt for helpful comments and suggestions. We are particularly grateful to Peter Wakker who sharpened our theoretical understanding of the experimental findings. Aylit and Nicky gratefully acknowledge financial funding of the experiment by the School of Economics and Business Sciences at Wits.

†Department of Economics, University of Pretoria, South Africa. E-mail: nicky.nicholls@up.ac.za

‡School of Economic and Business Sciences, University of the Witwatersrand, Johannesburg, South Africa. E-mail: aylit.romm@wits.ac.za

§Corresponding author. Department of Economics, University of Pretoria, Private Bag X20, Hatfield 0028, South Africa. Tel: 0027 12 420-3522. Fax: 0027 86 691-2749. E-mail: alexander.zimper@up.ac.za

1 Introduction

As a reaction to empirically observed violations of Savage's (1954) sure-thing principle (STP), a class of axiomatic decision theories had been developed which models any deviation from expected utility (EU) theory through ambiguity[1] attitudes without giving up von Neumann and Morgenstern's (1947) independence axiom (IA) for known probabilities. For example, the celebrated Choquet expected utility (CEU) model of Schmeidler (1989) expresses ambiguity attitudes through non-additive probability measures whereas the multiple priors expected utility (MEU) models of Gilboa and Schmeidler (1989) and Ghirardato et al. (2004) use sets of subjective additive probability measures. These highly influential models reduce to standard EU theory in the absence of ambiguity; i.e., whenever a non-additive probability measure reduces to an additive probability measure or a set of priors only contains a unique additive probability measure.

For an intuitive understanding of ambiguity the multiple priors approach has proved to be persuasive because it interprets ambiguity as a lack of statistical information. In the words of Gilboa and Schmeidler (1989, p. 142): "[...] the subject has too little information to form a prior. Hence (s)he considers a set of priors." By this interpretation, one would intuitively expect that a decision maker's ambiguity attitudes will become more and more irrelevant with statistical learning. We follow here Marinacci (2002) who formulates this intuition as follows:

> "Consider a decision maker (DM) who has to make a decision based on the drawings of an urn of known composition. The confidence he has in his decisions will depend on the quality of the information on the balls' proportion, the more he knows, the more he will feel confident.
>
> Suppose the DM can sample with replacement from this urn before making a decision. Regardless of how poor is his *a priori* information about the balls' proportions, it is natural to expect that eventually, as the number of observations increases, he will become closer and closer to learn the true balls' proportion and become more and more confident in his decisions." (p. 143)

Under the plausible notion that ambiguity decreases through statistical learning, an CEU as well as an MEU decision maker will ever closer resemble an EU decision maker whenever more and more statistical information becomes available. Consequently, for such decision makers we would conjecture a decline in the number of EU violations if they are exposed to ongoing statistical learning.

[1] Somewhat losely, we speak of "ambiguity" whenever a decision maker cannot comprehensivley resolve all his uncertainty through a unique additive–either subjective or objective–probability measure.

In this paper we report the results of an experimental study which has put this conjecture to the empirical test. In this experiment respondents were asked to choose between pairs of uncertain prospects such that the actual payout is determined by a ball's color drawn from an urn containing 20 red, 20 yellow and 60 blue balls whereby these proportions were not known by the respondents. In total there were 30 questions, organized as 15 consecutive pairs of questions, such that the answers to each consecutive pair of questions either violate the STP or not. After answering each question the respondents of the test group received statistical information in the form of a ball's color drawn from the urn. In contrast, the respondents of the control group did not receive any information as feedback. That is, the control group was asked the same questions in the same experimental situation with only the specific experimental treatment—an increase in statistical information—missing. Our research question whether EU violations decrease through statistical learning was thus addressed by the experiment as follows:

> Compared to the control group, do the STP violations of the test group decrease over the course of the experiment?

At first, we had to assess whether statistical learning, in the sense of subjective beliefs converging to objective probabilities, did actually happen within the test group or not. To this purpose we asked all respondents at the end of the experiment for their estimates about the differently colored balls' proportions. Whereas the control group—having received no statistical information—estimated a roughly equal proportion for all colors, the test group's estimate was very close to the true proportions. We interpret these answers as confirmation that converging statistical learning did indeed happen within the test group whereas the control group remained ignorant about the true proportions.

Next, turning to our experimental results we continue to see the same prevalence of STP violations in the test group even after multiple rounds of statistical learning. In contrast, a moderate decrease in STP violations is significant for the control group despite the fact that this group did not receive any statistical information. We therefore conclude that our experimental results do not support the notion that STP violations decrease through statistical learning.

To interpret our findings we adopt Peter Wakker's point of view that situations of decision making under risk should be regarded as limiting cases of situations of decision making under uncertainty (cf. Chapter 2 in Wakker 2010). Compared to decision making under uncertainty, which refers to situations where "objective/physical" probabilities of events are not known, decision making under risk refers to situations where people (believe to) know such probabilities. By decreasing ambiguity through statistical learning, our experiment gradually transforms an uncertain into a risky decision situation for the respondents of the test group. For a risky decision situation, our test of STP violations becomes observationally equivalent to a test for IA violations. To observe no decline in violations of the STP if ambiguity gradually vanishes therefore suggests the following limit relationship:

> Violations of the STP in situations under uncertainty carry over to violations of the IA when uncertain situations transform, through a decrease of ambiguity, into risky situations.

We conclude that empirically observed STP and IA violations are related phenomena so that a good descriptive decision theory should be able to accommodate violations of the STP as well as of the IA. This last point supports a similar conclusion by Peter Wakker and coauthors (cf., e.g., Wakker 2001; Abdellaoui et al. 2011). Namely, these authors promote a class of axiomatic decision theories—subsumed under the label *prospect theory* and encompassing Gilboa's (1989) CEU axiomatization as well as the axiomatizations of cumulative prospect theory in (Tversky and Kahneman 1992; Wakker and Tversky 1993; Tversky and Wakker 1995)—for the very reason that these models do not reduce to EU theory in the absence of ambiguity.

The remainder of this paper is structured as follows. Section 2 briefly reviews decision theoretic preliminaries. Section 3 discusses the experimental methodology used and Section 4 reports the results of the experiment. In Section 5 we discuss our findings and Section 6 concludes. We provide further details (instructions, questionnaire) of the experiment in a Supplementary Appendix.

2 Theoretical preliminaries: Ambiguity within the Anscombe Aumann (1963) framework

Savage's (1954) axiomatization of EU theory considers a state space S, a set of outcomes X, and a set of acts F which contains mappings from the state into the outcome space. Savage proceeds by providing a set of axioms which gives rise to the EU-representation of preferences $\succeq$ over acts such that, for all $f, g \in F$,

$$f \succeq g \Leftrightarrow \int_{s \in S} u(f(s))\, d\pi(s) \geq \int_{s \in S} u(g(s))\, d\pi(s) \tag{1}$$

for some bounded function $u : X \to \mathbb{R}$, unique up to a positive affine transformation, and a unique subjective additive probability measure π. Savage's key axiom for deriving the EU representation (1) is the sure-thing principle (STP) which states that for all $f, g, h, h' \in F$ and all events $E \subset S$,

$$f_E h \succeq g_E h \ \Rightarrow \ \ f_E h' \succeq g_E h' \tag{2}$$

where $f_E h$ denotes the act that gives the outcomes of act f in event E and the outcomes of act h else, i.e.,

$$f_E h(s) = \begin{cases} f(s) & \text{for } s \in E \\ h(s) & \text{for } s \in S \backslash E. \end{cases} \tag{3}$$

Already Ellsberg (1961) reports systematic violations of the STP in experimental situations. To accommodate Ellsberg paradoxes Schmeidler (1989), Gilboa and Schmeidler (1989), and Ghirardato et al. (2004) have axiomatized generalizations of EU theory within the Anscombe-Aumann (1963) (AA) framework, which considers decision makers who have preferences over known probability distributions. More precisely, AA axiomatize EU theory by adopting Savage's notion of preferences over acts (*horse lotteries*) but consider a set of outcomes $X = \Delta(Z)$ which is given as the set of all (finite) probability distributions (*roulette lotteries*) over a set of prizes Z. As a consequence, for any two acts $f, g \in F$, the mixture act

$$\lambda f + (1 - \lambda) g \text{ with } \lambda \in (0, 1) \tag{4}$$

is well-defined as the act in F that gives in any state $s \in S$ the reduced compound lottery $\lambda f(s) + (1 - \lambda) g(s)$ as outcome. Key to the AA axiomatization of EU theory is the *AA-independence axiom* which is defined as follows: For all $f, g, h \in F$ and all $\lambda \in (0, 1)$,

$$f \succeq g \Leftrightarrow \lambda f + (1 - \lambda) h \succeq \lambda g + (1 - \lambda) h. \tag{5}$$

The following observation is formally proved in the appendix.

Observation 1. *Within the AA framework, the AA-independence axiom (5) implies the STP (2).*

By Observation 1, any AA axiomatization which allows for STP violations must relax the AA-independence axiom (5). In a seminal contribution, David Schmeidler (1989) restricts the AA-independence axiom to the domain of all (pairwise) *comonotonic* acts $f, g, h \in F$ whereby the acts f and g are comonotonic iff, for all $s, t \in S$, $f(s) \succeq f(t)$ implies $g(s) \succeq g(t)$. Under this weakened version of the AA-independence axiom, Schmeidler (1989) derives the following Choquet expected utility (CEU) representation for preferences over all acts $f, g \in F$,

$$f \succeq g \Leftrightarrow \int_{s \in S}^{C} u(f(s))\, d\nu(s) \geq \int_{s \in S}^{C} u(g(s))\, d\nu(s) \tag{6}$$

where ν is a unique non-additive (=not necessarily additive) probability measure satisfying $\nu(\emptyset) = 0$, $\nu(S) = 1$, and $\nu(E) \leq \nu(E')$ if $E \subset E'$; and the integral in (6) is the Choquet integral.[2] Schmeidler (1989) proceeds to define *ambiguity aversion* as quasiconcave preferences in the following sense

$$f \succeq g \Rightarrow \lambda f + (1-\lambda) g \succeq g \text{ for all } \lambda \in (0,1). \tag{8}$$

This definition (roughly) implies that ambiguity averse decision makers like hedging in the form of combining acts with negatively correlated outcomes. Schmeidler (1989) proves that the CEU model (6) represents ambiguity averse preferences (8) in the AA framework if and only if the non-additive probability measure ν is *convex*.[3]

Also within the AA framework, Gilboa and Schmeidler (1989) impose ambiguity aversion (8) as an axiom. By restricting the domain of the AA-independence axiom (5) to all $f, g \in F$ and all constant $h \in F$ (called *certainty-independence*), the authors axiomatize the following multiple priors expected utility (MEU) representation of ambiguity averse preferences over all $f, g \in F$:

$$f \succeq g \Leftrightarrow \min_{\pi \in \Pi} \int_{s \in S} u(f(s))\, d\pi(s) \geq \min_{\pi \in \Pi} \int_{s \in S} u(g(s))\, d\pi(s) \tag{10}$$

for some unique (closed and convex) set of additive probability measures Π. According to this (maxmin) MEU representation, an ambiguity averse decision maker who considers several probability measures as possible chooses his acts under the assumption that a malign nature always picks the worst-case scenario probability measure that minimizes the expected utility of his chosen act. Note that Schmeidler's CEU representation (6) of preferences over acts with respect to a convex non-additive probability measure is formally equivalent to the MEU representation (10) if the set of priors is given as the core of this non-additive probability measure.[4]

[2] If f only takes on finitely many, say m, values, we have

$$\int_{s \in S}^{C} u(f(s))\, d\nu(s) = \sum_{i=1}^{m} u(f(s_i)) \cdot [\nu(E_1 \cup ... \cup E_i) - \nu(E_1 \cup ... \cup E_{i-1})] \tag{7}$$

where $E_1, ..., E_m$ denotes the unique partition of S with $u(f(s_1)) > ... > u(f(s_m))$ for $s_i \in E_i$.

[3] ν is convex iff it satisfies for all events E, E'

$$\nu(E \cup E') + \nu(E \cap E') \geq \nu(E) + \nu(E'). \tag{9}$$

[4] More precisely, recall that Schmeidler (1986) proves for convex ν that

$$\int_{s \in S}^{C} u(f(s))\, dv(s) = \min_{\pi \in \Pi} \int_{s \in S} u(f(s))\, d\pi(s) \tag{11}$$

Like Gilboa and Schmeidler (1989), Ghirardato et al. (2004) also weaken the AA-independence axiom by imposing certainty independence. However, in line with empirical evidence according to which real-life decision makers are not exclusively ambiguity averse (cf., e.g., Wu and Gonzalez 1999; Wakker 2010 and references therein), Ghirardato et al. (2004) axiomatize the following (maxmin-maxmax) MEU representation of preferences over acts within the AA framework (Theorem 11, p. 148):

$$\begin{aligned} f &\succeq g \Leftrightarrow \\ & \alpha(f) \cdot \min_{\pi \in \Pi} \int_{s \in S} u(f(s))\, d\pi(s) + (1 - \alpha(f)) \cdot \max_{\pi \in \Pi} \int_{s \in S} u(f(s))\, d\pi(s) \\ &\geq \alpha(g) \cdot \min_{\pi \in \Pi} \int_{s \in S} u(g(s))\, d\pi(s) + (1 - \alpha(g)) \cdot \max_{\pi \in \Pi} \int_{s \in S} u(g(s))\, d\pi(s) \end{aligned} \tag{13}$$

where the (unique) function α maps the set of acts into the unit interval. If $\alpha(f) = 1$ for all acts f, this representation obviously reduces to the original Gilboa and Schmeidler (1989) MEU model (10). More generally, Ghirardato et al. (2004) argue that (13) also encompasses any CEU representation (6) with respect to arbitrary non-additive probability measures.

Finally, observe that the CEU representation (6) becomes the EU representation (1) for an additive $\nu = \pi$ and that the MEU representations (10) and (13) become the EU representation (1) for an Π that only contains a single additive probability measure π. That is, all the above AA axiomatizations of ambiguity attitudes reduce to EU theory if ambiguity vanishes from the decision situation. In the remainder of the paper, we describe and discuss an experiment that has put this formal implication of the existing AA models of ambiguity attitudes to the empirical test.

3 Experiment

This section describes in detail the experiment that we conducted to test whether violations of the STP decline through statistical learning or not. Uncertainty was generated by means of an ambiguous urn containing balls of red, yellow, and blue color. Whereas the test group was given the opportunity to gradually learn the true proportions of differently colored balls in the urn, the control group did not receive any statistical information over the course of the experiment. The test for violations of the STP follows

---

such that

$$\Pi = core(\nu) \equiv \{\text{additive } \pi \mid \pi(s) \geq \nu(s) \text{ for all } s \in S\}. \tag{12}$$

similar approaches in the literature (e.g., Wu and Gonzalez 1999) whereby a caveat applies (see the Appendix).

3.1 Eliciting violations of the STP

Consider at first a standard (static) decision situation without statistical learning. Let us partition the state space S into the three events R(ed ball is drawn), Y(ellow ball is drawn), and B(lue ball is drawn). We test for violations of the STP through a choice between prospects **A** and **B**, on the one hand, and a choice between prospects **A**$'$ and **B**$'$, on the other hand. These pairs of prospects have the following payoff structure:

	R	Y	B
A	y	x	z
B	y	y	y

	R	Y	B
A$'$	x	x	z
B$'$	x	y	y

where $z > y > x$ are monetary amounts. In the experiment any such pair of prospects will correspond to a pair of observed choices whereby we interpret these choices as revealed (strict) preferences; that is, we interpret, e.g., the choice pair $\mathbf{A}, \mathbf{A}'$ as revealed preferences $\mathbf{A} \succ \mathbf{B}$ and $\mathbf{A}' \succ \mathbf{B}'$.

Obviously, the choice pairs

$$\mathbf{A}, \mathbf{A}' \text{ and } \mathbf{B}, \mathbf{B}' \tag{14}$$

are consistent with the STP, whereas the choice pairs

$$\mathbf{A}, \mathbf{B}' \text{ and } \mathbf{B}, \mathbf{A}' \tag{15}$$

violate the STP. To see this just reinterpret the prospects as the following acts in (2)

$$\mathbf{A} = f_{Y \cup B} h, \ \mathbf{B} = g_{Y \cup B} h; \tag{16}$$

$$\mathbf{A}' = f_{Y \cup B} h', \ \mathbf{B}' = g_{Y \cup B} h' \tag{17}$$

where $h(s) = y$ and $h'(s) = x$ for $s \in R$.

3.2 Subjects

Undergraduate Commerce students at the University of the Witwatersrand (Wits) were recruited to participate in the experiment. Students were approached during Economics lectures and were given an information sheet with brief background about the study and requesting their voluntary participation. Students agreeing to participate had to be over 18 years of age, and had to be students at Wits. Since the study design included a test and control group, the experiment was designed to start with a fairly homogenous sample in terms of education, then to randomly assign participants to one of the two groups. To avoid introducing possible biases in responses, the presence of the test and control groups was not discussed with participants, nor was the exact nature of the experiment. Participants were simply told that this was a study on decision making.

In total, 63 students were recruited, allowing for a minimum of 30 students in each group (31 in the test group and 32 in the control group). The control group had 56% male and 44% female participants, while the test group had 48% male and 52% female participants. In line with Wits' ethical policy on experiments, participants were assured of anonymity in the experiment whereby questionnaire responses were recorded with numbers instead of names.

3.3 Stimuli and statistical learning

We used an actual urn that contained 20 red, 20 yellow, and 60 blue balls. Although the participants did not know the true proportions of the colors in the urn, they were informed that there were 100 balls in total. All respondents answered 30 questions about their preferred choice between two different prospects. By varying the monetary payoffs associated with these prospects (measured in South African Rand)[5], these 30 questions were organized as 15 subsequent choice pairs $(\mathbf{A}, \mathbf{B}; \mathbf{A}', \mathbf{B}')$ exhibiting the payoff structure described in Section 3.1. This design thus allowed for the observation of up to 15 subsequent violations of the STP by any subject through revealed preferences (15).

Following the draw of a ball after each question, respondents in the test group received feedback on the color of the ball drawn from the urn. Respondents from the control group did not receive such feedback. In contrast to the control group, the test group could thus observe statistical information in the form of 30 actual drawings (with replacement) from the urn. Technically speaking, the respondents from the test group were thus exposed to 30 successive i.i.d. multivariate Bernoulli trials such that the true proportions of differently colored balls were (supposedly) driving the data generation process.

3.4 Procedure

The respondents sat in front of the computer where they first read participant instructions and then went through the 30 questions of the questionnaire (see the Supplementary

[5]While we fixed the worst payoff x at zero, the second best payoff y was in the range of R40 to R80, and the best payoff z was in the range of R75 to R150. The Supplementary Appendix lists the questionnnaire containing all 30 questions.

Appendix). Nicky was present at all times to answer questions and to assist with the random ball selection following each question for the test group.

We used randomization of the question pair order to avoid any bias from possible order effects due to, e.g., different magnitudes or ratios from the prospects' possible payoffs. More specifically, the computer programme selected a random starting question pair, and randomized each subsequent question pair. In this way, each respondent would see a unique order of questions, with all respondents seeing all 15 pairs of questions, but in a random order. To avoid bias from the order of presentation within question pairs the labelling of prospects within question pairs was varied.

Once an option had been selected, the test group respondents were allowed to draw a ball from the urn and the computer programme would show the payout based on the color of ball drawn and the prospect selected. The control group received no feedback.

To give incentives for the truthful revelation of preferences, subjects in both groups were told that one of the prospect choices would be selected at random to be paid out in real money (cf., e.g., Starmer and Sugden 1991). The possible payout could be as low as R0 or as high as R150, since these were the minimum and maximum payouts across the range of questions.

At the end of the experiment, i.e. after answering all 30 questions and receiving (the test group) versus not receiving (the control group) statistical information, the respondents from both groups were asked about their estimates for the balls' proportions in the urn.

4 Results

4.1 Statistical learning

We could confirm that statistical learning did indeed happen within the test group. According to standard models of statistical learning, subjective estimates converge to the true proportions of differently colored balls (='objective' probabilities) if the respondents can observe large data samples from multivariate Bernoulli trials (cf., e.g., Viscusi and O'Connor 1984; Viscusi 1985; Chapter 4 and references in Zimper 2013). Answers close to the true proportions of 60% blue, 20% red, and 20% yellow balls in the test group would thus suggest that statistical learning happened within the test group. In contrast, given the absence of any statistical information, the control group was expected to report roughly equal proportions of all colors as an expression of their ignorance.

Table 1: Average estimates for the proportions of differently colored balls

	% Red	% Yellow	% Blue
Control group	33	31	36
Test group	25	21	54

Both patterns are indeed reflected by the reported proportions (cf. Table 1); for example, the average estimate of the test group for the proportion of blue balls was 54% compared to 36% average estimate of the control group. We therefore conclude that exposing the test group to the outcomes of 30 multivariate Bernoulli trials was indeed sufficient to distinguish the test from the control group with respect to statistical learning.

4.2 Revealed choices

Table 2 below presents the percentage of choice pairs seen in the control and test group, respectively. Since the question order was randomized to avoid order biases, these question pairs refer to the first, second, etc. pair of questions seen by each respondent. That is, each respondent from the test group had observed 28 subsequent drawings (with replacement) of balls before he/she answered the 15th (=final) question pair.

Table 2: Revealed choices (in %) over rounds of questions

	Control Group				Test Group			
Question pair	$\mathbf{A}, \mathbf{A}'$	$\mathbf{B}, \mathbf{B}'$	$\mathbf{A}, \mathbf{B}'$	$\mathbf{B}, \mathbf{A}'$	$\mathbf{A}, \mathbf{A}'$	$\mathbf{B}, \mathbf{B}'$	$\mathbf{A}, \mathbf{B}'$	$\mathbf{B}, \mathbf{A}'$
1	6	72	16	6	10	58	16	16
2	6	69	19	6	6	48	39	6
3	6	72	9	13	16	48	23	13
4	6	56	19	19	13	42	39	6
5	6	69	19	6	13	45	32	10
6	6	78	16	0	10	35	52	3
7	6	78	13	3	13	42	35	10
8	9	59	25	6	19	45	32	3
9	6	72	22	0	16	39	32	13
10	9	75	16	0	16	45	35	3
11	9	81	9	0	10	42	35	13
12	13	75	9	3	13	45	35	6
13	9	75	16	0	16	45	32	6
14	3	75	22	0	23	29	35	13
15	9	81	6	3	23	55	16	6

4.3 STP violations

Recall that the choice pairs $\mathbf{A}, \mathbf{A}'$ and $\mathbf{B}, \mathbf{B}'$ are consistent with the STP whereas the choice pair $\mathbf{A}, \mathbf{B}'$ and $\mathbf{B}, \mathbf{A}'$ are not. According to Table 2, the majority of respondents in both groups do not violate the STP; whereby we put down the higher proportion of STP consistent responses in the control group to random preference differences. Among the STP violating responses in both groups, the majority chooses $\mathbf{A}, \mathbf{B}'$.

Turn now to our question whether STP violations decrease through statistical learning or not. If this was the case, overall responses should show decreased STP violations for the test group over the course of the experiment but little change in the control group. By the chart in Figure 1, STP violations do not show any clear downward trend for the test group whereas there is a mild downward trend for the control group (albeit with some noise).

To obtain a more precise picture, we start by estimating the following equation for our group of respondents as a whole:

$$P = \beta_0 + \beta_1 D + \beta_2 D Round + \beta_3 (1 - D) \, Round + \varepsilon \tag{18}$$

where P denotes the proportion (in decimals) of respondents violating the STP, *Round* stands for the number of question pairs answered, and $\varepsilon \sim N(0, \sigma^2)$. We set the dummy variable $D = 1$ for the test and $D = 0$ for the control group so that β_1 measures

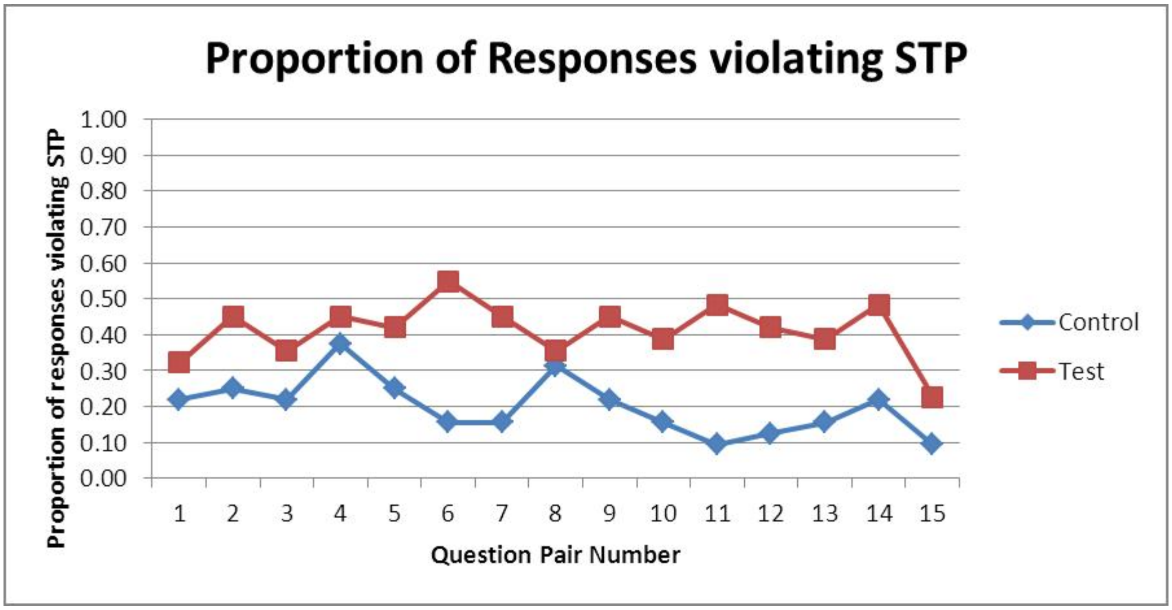

Figure 1: Violations of the STP over the course of the experiment

the difference between the test and the control group, β_2 measures the impact on STP violations of an increase in the number of question pairs for the test group, whereas β_3 measures the same impact for the control group. In particular we estimate both an OLS regression, and an iterated reweighted least squares regression (IRLS). An IRLS regression is very much like an OLS regression, except for the fact that it accounts for and reduces the influence of extreme observations (observations with large residuals). It thus produces robust estimates. The results are reported in Table 4.

Table 4: Estimates of the OLS and IRLS regressions

	β_0	β_1	β_2	β_3
OLS	0.28*(0.004)	0.14* (0.057)	-0.002(0.004)	-0.01*(0.004)
IRLS	0.27*(0.004)	0.13*(0.057)	0.002(0.004)	-0.009*(0.004)

Notes: standard errors in parentheses
* significant at the 5% level

The results for the OLS and IRLS regressions are very similar. The significant positive coefficient β_1 on the test group dummy shows that there are fewer STP violations in the control than in the test group. For the IRLS regression, in particular, the coefficient implies that the test group is violating the STP by 13 percentage points more than the control group. The estimated β_2 coefficient is very small and insignificant suggesting that the number of experimental rounds for the test group has zero impact on STP violations. In contrast, the negative coefficient β_3, which measures changes in STP violations over rounds for the control group, is significant. In particular, the decrease in STP violations is about 1 percentage point per round for the control group.

In addition, we conducted a maximum likelihood estimation in the form of logit regression for our individual respondents. The coefficients of a logit regression represent the change in the log of the odds associated with a unit change in the explanatory variable. For ease of interpretation, we also report the marginal effects (ME), which show the increase in the probability, at the means of the covariates, of an individual violating the STP. We conduct both a logit regression on the pooled sample, and a fixed effects logit, which allows us to control for idiosyncratic characteristics of the individual that is fixed across rounds, yet might influence his/her probability of violating the STP. We report these results in table 5.

Table 5: Logit Estimates

	$\boldsymbol{\beta}_0$	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_3$
POOLED LOGIT	0.91*(0.227)	0.63*(0.299)	-0.01(0.021)	-0.063*(0.0044)
ME POOLED LOGIT		0.131*	-0.002	-0.01*
FIXED EFFECTS LOGIT			-0.01(0.024)	-0.082*(0.031)
ME FIXED EFFECTS LOGIT			-0.002	-0.02*

These regressions confirm the fact that the number of experimental rounds has a basically zero effect on the probability of an individual in the test group violating the STP. Again, the pooled logit confirms that the probability of an individual in the test group violating the STP is 13 percentage points greater than that of an individual in the control group. We also confirm that the probability of violating the STP decreases over rounds for the control group, with the effect being stronger when we control for individual fixed effects. In particular the fixed effects logit predicts that with every round, the probability of an individual in the control group violating the STP, decreases by 2 percentage points.[6]

Since the control group did not receive any statistical information after making their choices, our set-up does not offer any apparent explanation for this small but significant decrease in STP violations except for the "greater experience" that comes from more

[6]Note that there is no $\boldsymbol{\beta}_1$ coefficient for the fixed effects logit since the dummy variable, D, representing whether an individual is in the test or control group is fixed over rounds, and is thus dropped from the regression.

rounds of making choices. If such "greater experience" also had any positive impact on the test group's STP consistency, this impact was off-set by some negative impact, which we can only attribute to the test group's treatment in the form of statistical learning.

To sum up: Our experimental data does not lend any support to the notion that STP violations decrease through statistical learning.

5 Discussion

Because existing AA axiomatizations of ambiguity attitudes predict a decline in STP violations when ambiguity vanishes from the decision situation, they cannot explain our data. To interpret this descriptive shortcoming we first demonstrate that the axiomatic models of Schmeidler (1989), Gilboa and Schmeidler (1989), and Ghirardato et al. (2004) imply the validness of von Neumann and Morgenstern's (1947) independence axiom (IA) despite their weakening of the AA independence axiom.

In a second step, we argue that our empirical test for violations of the STP, in the form of observed choice pairs $\mathbf{A}, \mathbf{B}'$ or $\mathbf{B}, \mathbf{A}'$, becomes a test for violations of the IA whenever the decision maker resolves his uncertainty through a unique additive probability measure in the limit of the statistical learning process. The fact that we do not observe a decline of STP violations in the test group might then be naturally attributed to the well known fact that violations of the IA are a common empirical phenomenon for decision making under risk (cf., e.g., Allais 1979; Wu and Gonzalez 1996; Starmer 2000; Schmidt 2004; Sugden 2004).

5.1 The von Neumann and Morgenstern independence axiom in an AA framework

Let us follow Peter Wakker and regard decision making under risk as the limiting case of decision making under uncertainty:

> "Probabilities can be (un)known to many degrees, all covered by the general term uncertainty. Decision under risk is the special, limiting case where probabilities are objectively given, known, and commonly agreed upon." (Wakker 2010, p. 44)

In contrast to theories of decision making under uncertainty, theories of decision making under risk fix some probability space, say $(S, \pm, \pi^*)$ (with $\pm$ denoting some σ-algebra on S), and consider a decision maker with preferences $\geq$ over *lotteries* in $\Delta(Z)$ where $Z \subset \mathbb{R}$ denotes a set of deterministic prizes. Formally, these lotteries are

$\pm$-measurable random variables on S with finite support on Z whose distribution is governed by π^*.

Note that there exists a one-one correspondence between all constant acts in the AA framework, i.e., all acts which give in any state of the world the same lottery in $\Delta(Z)$ as outcome, and all lotteries $\Delta(Z)$ in the vNM framework. Consequently, there exists an equivalence relation between (revealed) AA preferences $\succeq$ restricted to constant acts with outcomes in $\Delta(Z)$, on the one hand, and (revealed) vNM preferences $\geq$ defined over the lotteries in $\Delta(Z)$, on the other hand; which we can formally express as follows: for all $L, L' \in \Delta(Z)$,

$$L \geq L' \Leftrightarrow f^L \succeq f^{L'}, \tag{19}$$

where f^L is the act that gives in every state $s \in S$ the lottery L as outcome.

EU theory had been first axiomatized within the framework of decision making under risk by von Neumann and Morgenstern (vNM) (1947). Key to their celebrated EU representation theorem is the *independence axiom* (IA) which implies that, for all $L, L', L'' \in \Delta(Z)$ and all $\lambda \in (0,1)$,

$$L \geq L' \Leftrightarrow \lambda \cdot L + (1-\lambda) L'' \geq \lambda \cdot L' + (1-\lambda) L'' \tag{20}$$

(cf., e.g., Fishburn 1988). Observe that Schmeidler's (1989) comonotonic independence axiom as well as Gilboa and Schmeidler's (1989) and Ghirardato et al.'s (2004) certainty independence axiom imply that the AA independence axiom (5) holds for all constant acts. That is, for all $L, L', L'' \in \Delta(Z)$ and all $\lambda \in (0,1)$,

$$f^L \succeq f^{L'} \Leftrightarrow \lambda \cdot f^L + (1-\lambda) f^{L''} \succeq \lambda \cdot f^{L'} + (1-\lambda) f^{L''}, \tag{21}$$

which is, by (19), equivalent to the IA (20). The following observation summarizes this relationship which is the formal reason for the fact that existing AA axiomatizations of ambiguity attitudes reduce to EU theory whenever ambiguity vanishes from the decision situation.

Observation 2. *All weakened versions of the AA-independence axiom (5) in the existing AA axiomatizations of ambiguity attitudes imply the vNM independence axiom (20).*

5.2 Reinterpreting violations of STP after statistical learning

Turn now back to our experiment and observe that it gradually decreases ambiguity for the test group whereby the limiting case corresponds to the situation in which the decision maker could observe an infinite amount of drawings (with replacement) from the urn, i.e., an infinite amount of data generated by multivariate Bernoulli trials driven by the balls' true proportions. According to standard models of statistical Bayesian learning, this decision maker will then learn, by Doob's (1949) consistency theorem, with certainty the true probabilities of the events R, Y, and B.[7]

Denote this objective probability measure as π^*; e.g., in our experiment we stipulate that $\pi^*(R) = \pi^*(Y) = 0.2$, $\pi^*(B) = 0.6$ in accordance with the balls' true proportions. In the limit of the statistical learning process, the uncertain decision situation would thus be transformed into a decision situation under risk such that the prospects **A** and **B** with, e.g., payoff matrix

	R	Y	B
A	y	x	z
B	y	y	y

can be interpreted as acts in the AA framework such that

$$\mathbf{A} \sim f^L, \quad (22)$$

$$\mathbf{B} \sim f^{L'} \quad (23)$$

where f^L and $f^{L'}$ are constant acts with outcomes in $\Delta\{x, y, z\}$ such that

$$L = (0.2, 0.2, 0.6), \quad (24)$$

$$L' = (0, 1, 0). \quad (25)$$

That is, in the limit the uncertain prospect **A** has become equivalent (with respect to the decision maker's AA preferences) to the risky lottery (i.e., random variable) which gives prizes x with 20% chance, y with 20% chance, and z with 60% chance.

Further note that the choice pairs $\mathbf{A}, \mathbf{B}'$and $\mathbf{B}, \mathbf{A}'$, which revealed violations of the STP under uncertainty, reveal in the risky limit situations violations of IA (20). To see this denote by δ_k, $k \in \{x, y, z\}$, the degenerate lottery that gives amount k with probability one. Let $\lambda = \pi^*(R)$ and $(1-\lambda) \cdot \mu = \pi^*(Y)$ and observe that (20) implies for the choice pair $\mathbf{A}, \mathbf{B}'$ that[8]

$$\mathbf{A} \succ \mathbf{B} \Leftrightarrow \quad (26)$$

$$\lambda \cdot \delta_y + (1-\lambda) \cdot [\mu \cdot \delta_x + (1-\mu) \cdot \delta_z] > \succ \lambda \cdot \delta_y + (1-\lambda) \cdot \delta_y \Leftrightarrow \quad (27)$$

$$\mu \cdot \delta_x + (1-\mu) \cdot \delta_z > \delta_y. \quad (28)$$

[7]For a more detailed discussion of the relationship between converging Bayesian learning and Doob's consistency theorem see, e.g., Diaconis and Freedman (1986), Chapter 1 in Gosh and Ramamoorthi (2003), Lijoi, Pruenster and Walker (2004), or Chapter 4 in Zimper (2013).

[8]We assume that the standard principle of *reduction of compound lotteries* applies (cf., e.g., Fishburn 1988).

as well as

$$\begin{aligned} \mathbf{B}' &\succ \mathbf{A}' \Leftrightarrow & (29) \\ \lambda \cdot \delta_x + (1-\lambda) \cdot \delta_y &> \lambda \cdot \delta_x + (1-\lambda) \cdot [\mu \cdot \delta_x + (1-\mu) \cdot \delta_z] \Leftrightarrow & (30) \\ \delta_y &> \mu \cdot \delta_x + (1-\mu) \cdot \delta_z. & (31) \end{aligned}$$

Since (28) and (31) constitute a contradiction, the revealed preferences $\mathbf{A}, \mathbf{B}'$ violate the IA.

To conclude this discussion: Our experiment suggests that existing AA axiomatizations of ambiguity attitudes cannot explain the observed data because they do not account for violations of the IA whenever ambiguity vanishes from the decision situation.

6 Concluding remarks

Existing AA axiomatizations of ambiguity attitudes reduce to EU theory whenever ambiguity vanishes from the model. We have conducted an experiment which gradually transforms a situation of decision making under uncertainty into a situation of decision making under risk. For our experimental environment, these models would therefore predict a decline of STP violations to the effect that the respondents of the test group should ever closer resemble an EU decision maker whenever ambiguity decreases through statistical learning over the course of the experiment. The findings of our experimental study, however, do not support the notion that such a decline occurs.

We were somewhat surprised that we did not observe, compared to the control group, any significant decline in choice patterns (15) for the test group because our experimental design seemed to be rigged towards such a comparative decline for two reasons. First, according to our caveat we cannot rule out that a decrease in choice patterns (15) for the test group was not exclusively caused by a decrease in STP violations but in addition by the update dynamics of EU consistent decision making for which our experiment did not control. That is, in contrast to the control group, converging statistical learning of EU decision makers in the test group could also have caused a decline in choice patterns (15).

Second, van de Kuilen and Wakker (2006) observe within a framework of decision making under risk a decrease in violations of the independence axiom if respondents repeatedly experience the consequences of their decisions in terms of played out lotteries. Again, in our experiment the respondents of the test but not of the control group were exposed to such experiences in terms of drawn balls. Suppose that some test group respondents became already very confident after a few rounds of statistical learning so that the observed violations of the STP can be interpreted as violations of the independence axiom. Then the findings of van de Kuilen and Wakker (2006) suggest that violations of the independence axiom should eventually decrease if these respondents go in subsequent rounds through the (non-statistical) learning experience documented by these authors. That is, we might have observed a decline in revealed choices (15) which would not be caused by statistical learning but rather by the increased experience in the risky environment in the sense of van de Kuilen and Wakker (2006).

The fact that neither the possible caveat nor the van de Kuilen and Wakker (2006) effect caused a stronger decline in choice patterns (15) in the test than in the control group can be interpreted as support for the notion that statistical learning does not reduce STP violations. Nevertheless, it would be desirable to run a similar but more sophisticated experiment which is able to control for both effects. In such an experiment, we would also try to better assess the subjects' probability assessments as well as their confidence in these assessments over the course of the experiment.

References

Abdellaoui, M., Baillon, A., Placido, L., and Wakker, P. P. (2011) "The Rich Domain of Uncertainty: Source Functions and their Experimental Implementation," *American Economic Review* **101**, 695–723.

Allais, M. (1979) "The Foundation of a Positive Theory of Choice Involving Risk and a Criticism of the Postulates and Axioms of the American School". Part II in Allais, M., and O. Hagen [eds.], *Expected Utility Hypotheses and the Allais Paradox*, D. Reidel: Dordrecht.

Anscombe, F. J., and R. J. Aumann (1963) "A definition of subjective probability," *Annals of Mathematical Statistics* **34**, 199-205.

Diaconis, P., and D. Freedman (1986) "On the Consistency of Bayes Estimates," *Annals of Statistics* **14**, 1-26.

Doob, J. L. (1949) "Application of the Theory of Martingales," In: *Le Calcul des Probabilites et ses Applications*, Colloques Internationaux du Centre National de la Recherche Scientifique 13, pp. 23–27. Paris: CNRS.

Ellsberg, D. (1961) "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics* **75**, 643-669.

Fishburn, P. C. (1988) *Nonlinear Preferences and Utility Theory under Risk*, The John Hopkins University Press: Baltimore, London.

Ghirardato, P. and M. Marinacci (2002), "Ambiguity Made Precise: A Comparative Foundation," *Journal of Economic Theory* **102**, 251-289.

Ghirardato, P., Maccheroni, F., and M. Marinacci, M. (2004) "Differentiating Ambiguity and Ambiguity Attitude," *Journal of Economic Theory* **118**, 133–173.

Gilboa, I. (1987) "Expected Utility with Purely Subjective Non-Additive Probabilities," *Journal of Mathematical Economics* **16**, 65-88.

Gilboa, I., and D. Schmeidler (1989) "Maxmin Expected Utility with Non-Unique Priors," *Journal of Mathematical Economics* **18**, 141-153.

Gosh, J. K., and R. V. Ramamoorthi (2003) *Bayesian Nonparametrics*, Springer Verlag Inc.: New York.

Lijoi, A., Pruenster, I., and S.O. Walker (2004), “Extending Doob’s Consistency Theorem to Nonparametric Densities,” *Bernoulli*, **10**, 651-663.

Marinacci, M. (2002) “Learning from Ambiguous Urns,” *Statistical Papers* **43**, 145-151.

Quiggin, J. P. (1981) “Risk Perception and Risk Aversion among Australian Farmers,” *Australian Journal of Agricultural Economics* **25**, 160-169.

Quiggin, J. P. (1982) “A Theory of Anticipated Utility,” *Journal of Economic Behavior and Organization* **3**, 323-343.

Savage, L. J. (1954) *The Foundations of Statistics*, John Wiley and & Sons, Inc.: New York, London, Sydney.

Schmeidler, D. (1986) “Integral Representation without Additivity,” *Proceedings of the American Mathematical Society* **97**, 255-261.

Schmeidler, D. (1989) “Subjective Probability and Expected Utility without Additivity,” *Econometrica* **57**, 571-587.

Schmidt, U. (2004) “Alternatives to Expected Utility: Some Formal Theories,” in: P.J. Hammond, S. Barberá, and C. Seidl [eds.], *Handbook of Utility Theory* Vol. II, Kluwer, Boston, Chapter 15.

Starmer, C. (2000) “Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk,” *Journal of Economic Literature* **38**, 332-382.

Starmer, C., and R. Sugden (1991) “Does the Random-Lottery Incentive System Elicit True Preferences? An Experimental Investigation,” *American Economic Review* **81**, 971–978.

Sugden, R. (2004) “Alternatives to Expected Utility: Foundation,” in: P.J. Hammond, S. Barberá, and C. Seidl [eds.], *Handbook of Utility Theory* Vol. II, Kluwer, Boston, Chapter 14.

Tversky, A., and D. Kahneman (1992) “Advances in Prospect Theory: Cumulative Representations of Uncertainty,” *Journal of Risk and Uncertainty* **5**, 297-323.

Tversky, A., and P. P. Wakker (1995) “Risk Attitudes and Decision Weights,” *Econometrica* **63**, 1255-1280.

Viscusi, W. K. (1985) “A Bayesian Perspective on Biases in Risk Perception,” *Economics Letters* **17**, 59-62.

Viscusi, W. K. and C. J. O'Connor (1984) "Adaptive Responses to Chemical Labeling: Are Workers Bayesian Decision Makers?," *American Economic Review* **74**, 942-956.

van de Kuilen, G., and P. P. Wakker (2006) "Learning in the Allais Paradox," *Journal of Risk and Uncertainty* **33**, 155-164.

von Neuman, J., and O. Morgenstern (1947) *Theory of Games and Economic Behavior*, Princeton University Press, 2nd edn., Princeton.

Wakker, P. P, and A. Tversky (1993) "An Axiomatization of Cumulative Prospect Theory," *Journal of Risk and Uncertainty* **7**, 147-167.

Wakker, P. P (2001) "Testing and Characterizing Properties of Nonadditive Measures through Violations of the Sure-Thing Principle," *Econometrica* **69**, 1039-1059.

Wakker, P. P (2010), *Prospect Theory for Risk and Ambiguity*, Cambridge University Press.

Wu, G., and R. Gonzalez (1996) "Curvature of the Probability Weighting Function," *Management Science* **42**, 1676-1690.

Wu, G., and R. Gonzalez (1999) "Nonlinear Decison Weighst in Choice under Uncertainty," *Management Science* **45**, 74-85.

Zimper, A. (2013) "The Emergence of "Fifty-Fifty" Probability Judgments through Bayesian Updating under Ambiguity," *Fuzzy Sets and Systems* **223**, 72-88.

Appendix

A1. Proof of Observation 1

Observe at first that the AA axiom *reversal of order in compound lotteries* implies, for all $f, g, h, h' \in F$ and all $E \subset S$,

$$\frac{1}{2}f_E h + \frac{1}{2}g_E h' \sim \left(\frac{1}{2}f + \frac{1}{2}g\right)_E \left(\frac{1}{2}h + \frac{1}{2}h'\right) \sim \frac{1}{2}f_E h' + \frac{1}{2}g_E h.$$

Now suppose to the contrary that $f_E h \succeq g_E h$ but $g_E h' \succ f_E h'$. Because of

$$\begin{aligned}
\frac{1}{2}f_E h + \frac{1}{2}f_E h' &\succeq \frac{1}{2}g_E h + \frac{1}{2}f_E h' \\
&\sim \left(\frac{1}{2}g + \frac{1}{2}f\right)_E \left(\frac{1}{2}h + \frac{1}{2}h'\right) \\
&\sim \frac{1}{2}f_E h + \frac{1}{2}g_E h'
\end{aligned}$$

(5) implies $f_E h' \succeq g_E h'$, a contradiction.□

A2. Caveat

Whereas the choice pairs (15) unambiguously reveal a violation of the STP for the control group, the situation is slightly different for the test group whose members observe after each choice the drawing of one ball. To see this consider a decision maker of the test group and denote by I_n the information he has received by observing n drawings. If this decision maker chooses, e.g., $\mathbf{A}$ after n and $\mathbf{B}'$ after $n+1$ drawings, these revealed choices could be rationalized as EU consistent choices as follows:

$$\mathbf{A} \succ \mathbf{B} \quad \Leftrightarrow \tag{32}$$

$$u(x) \cdot \pi(Y \mid I_n) + u(z) \cdot \pi(B \mid I_n) \quad > \quad u(y) \cdot (1 - \pi(R \mid I_n)) \tag{33}$$

and

$$\mathbf{B}' \quad \succ \quad \mathbf{A}' \Leftrightarrow \tag{34}$$

$$u(y) \cdot (1 - \pi(R \mid I_{n+1})) \quad > \quad u(x) \cdot \pi(Y \mid I_{n+1}) + u(z) \cdot \pi(B \mid I_{n+1}). \tag{35}$$

Whereas the two inequalities (33) and (35) cannot simultaneously hold if $\pi(\cdot \mid I_n)$ and $\pi(\cdot \mid I_{n+1})$ are sufficiently similar probability measures, it is possible that (33) and (35) are satisfied for appropiately chosen values of $\pi(\cdot \mid I_n)$ and $\pi(\cdot \mid I_{n+1})$. In that case, the choices $\mathbf{A}, \mathbf{B}'$ would not reveal a violation of the STP but rather the different perception of the urn's uncertainty by an EU decision maker before versus after he updates his additive belief on the $n+1$th observation.

We could have easily avoided this ambiguity in the interpretation of choice pairs (15) for the test group, if we had allowed for statistical learning not within but only after each question pair was answered. However, when designing the experiment, our concern was to give the subjects no hint that they were actually answering 15 well-structured question pairs rather than 30 similar questions. Whereas we feared that a detection of this question pair structure might eventually result in some answering bias, we assumed that any difference between $\pi(\cdot \mid I_n)$ and $\pi(\cdot \mid I_{n+1})$ would be negligibly small (in the sense of: How great can the impact of a single observation be?).

In the subsequent interpretation of the data, we therefore assume that $\pi(\cdot \mid I_n)$ and $\pi(\cdot \mid I_{n+1})$ are indeed sufficiently close so that choice pairs (15) cannot be explained by EU consistent decision making but rather by revealed violations of the STP. If this assumption was violated, however, we would observe, by Doob's (1949) consistency theorem, that $\pi(\cdot \mid I_n)$ and $\pi(\cdot \mid I_{n+1})$ become more and more similar with increasing n. Applied to the experiment, this means that although revealed choices (15) in the first rounds of the experiment (small n) might be EU consistent, revealed choices (15) in the later rounds of the experiment (large n) would rather indicate a violation of the STP. Consequently, an EU consistent decision maker who expressed choice pairs (15) in the beginning of the experiment, would eventually switch to choice pairs (14) in the later rounds of the experiment when his conditional probability measures converge through statistical learning in accordance with Doob's (1949) consistency theorem.

To summarize the caveat: If our maintained assumption that choice pairs (15) always reveal violations of the STP for the test group was not correct, we might observe a decline in the number of choice pairs (15) which is not caused by a decline of violations of the STP.