**Using Modern Portfolio Theory to Derive an Efficient Frontier of the South African Tax Product Mix**

Wian Boonzaaier[1]
South African Revenue Service[2]

ESSA 2013 Conference

**Abstract**

This paper will attempt to apply the mean-semi-variance optimisation framework to the current South African tax revenue structure. Mean-semi-variance optimisation aims to find the set of optimal weights allocated to each tax category that would minimise the downside volatility of the overall tax portfolio at a given level of portfolio growth, subject to certain constraints. The rationale behind mean-semi-variance optimization from a tax policy perspective is to derive a set of tax category combinations that would illustrate the trade-offs faced by the policymaker given the characteristics of each tax category and the interaction of these characteristics in a portfolio context. Using tax revenue data adjusted for inflation and discretionary changes over the period 1994/95 to 2012/13, preliminary results show that the current South African tax revenue structure lies slightly below the efficient frontier, while there is little scope for diversification in terms of reducing downside volatility. Personal income tax tends to dominate the efficient set of portfolios and also exhibits the most favourable growth-variability trade-off. Equity and efficiency was not considered given data limitations, and will be the subject of future research.

JEL Codes: H21 – Taxation and Subsidies: Efficiency; Optimal Taxation

Keywords: Modern portfolio theory; efficient frontier; optimal taxation; tax policy

[1] Corresponding author. Economic Research & Modelling Unit, Revenue Analysis Division, South African Revenue Service, Private Bag X923, Pretoria, 0001. E-Mail: WBoonzaaier@sars.gov.za.
[2] The views expressed herein are those of the author, and does not necessarily reflect those of the South African Revenue Service.

**Contents**

# 1. Introduction

Modern portfolio theory as formalised in Markowitz (1952) and Markowitz (1959) provided a framework for investors to select a portfolio of risky shares based on the trade-off between the risk and expected return of the shares. An important insight from this framework was the introduction of a quantitative measure of diversification in the form of the covariance between individual assets. The amount of diversification will be determined by the extent to which the assets in the investment portfolio is correlated, where lower correlation will result in lower overall risk of the investment portfolio at a given level of return. The application of modern portfolio theory has been largely confined to investment portfolios, notwithstanding its attractive optimisation framework for fields outside of the investment management sphere.

Optimal taxation theory, while useful in providing theoretical insights for designing a tax system, tend to be highly complex and abstract in nature. While it is recognised that the traditional objectives of tax policy focus on equity, efficiency and adequacy, factors such as revenue diversification and stability have also been viewed as desirable attributes of a good tax structure (Suyderhoud 1994: 168-171). Business cycle movements exert a significant impact on the growth and variability of tax revenue collections, typically in the form of budget over-collection during periods of economic expansion and revenue shortfalls during downswing phases of the economy (Schunk & Porca 2005: 99). The particular composition or structure of the tax portfolio may also influence its relative instability and growth potential. Although cyclical variability in revenue collections remains unavoidable to a large extent, it may be informative from a policy point of view to analyse the growth and variability characteristics of the individual tax products as well as the overall portfolio in order to better understand the determinants of revenue instability and the inherent trade-offs faced by policy makers.

The focus of this paper will be to use modern portfolio theory to derive an efficient frontier of the South African tax revenue mix, with the ultimate aim of illustrating the inherent trade-offs faced by policy makers in terms of tax revenue growth and variability. The preference function of policy makers will be left unspecified to ensure generality in the conclusions. To the best of the author’s knowledge, no study of this nature has been conducted for the case of South Africa.

## 2. Literature review

According to Markowitz (1952) and Markowitz (1959), the choice of securities to include in an investment portfolio will firstly involve a trade-off between risk and expected return. Expected return was defined as the expected change in the price of some asset over the measured period including any other form of income such as dividend payments, divided by the asset's beginning value. The suggested measure of risk was the dispersion or variance of the returns. Although not the first to consider the desirable characteristics of diversification, the research by Harry Markowitz was the first attempt to quantify the concept of diversification as it related to investment portfolios. It was also shown that through diversification, the risk of an investment portfolio can be significantly reduced without compromising the portfolio's expected return potential. This framework can also be applied to tax revenue portfolios, which will be the focus of the remainder of the literature review.

The degree of stability in a tax system was first addressed formally by Groves & Kahn (1952: 87-88), who characterised stability as a special case of adequacy. Adequacy, although not forming part of the maxims of taxation[3] as outlined by Smith (1776), refers to the potential of a tax product to produce a given amount of revenue and to sustain this level in a way that would maintain a given level of goods and services supplied by government. A stable tax system was therefore defined as one where real income to government over a period of time remains roughly constant. This definition implies that in an inflationary economic environment, the growth in tax revenue would need to change in proportion to growth in its tax base if the initial level of government goods wants to be maintained, assuming income changes in proportion to inflation. In contrast, when income changes in response to a change in real output, assuming a constant price level, the volume of government goods can be maintained without a change in tax yields.

One of the shortcomings of the approach followed by Groves & Kahn (1952) was their focus on the long run income elasticity of a tax product as an indicator of both its growth and instability characteristics. This may lead to the interpretation that tax products with high income elasticity will grow fast relative to its tax base and at the same time will be highly unstable, which is not necessarily the case. This aspect was partly addressed by Williams *et al* (1973) whereby the stability, growth and counter-cyclical characteristics of tax products were

[3] Adam Smith identified four principles of taxation: (1) Individuals should contribute towards government in proportion to their respective abilities (2) The timeline for payment, the manner in which it should be paid and the quantity should be certain (3) Taxes should be levied at the time or in a manner which would be convenient for the taxpayer (4) Taxes should be designed so that it does not burden taxpayers unnecessarily.

conceptually distinguished along with a quantitative measure for each. Building on this foundation, the paper by White (1983) attempted to explicitly incorporate the principles of portfolio selection as outlined in Markowitz (1952) by combining the growth and instability characteristics of state and local taxes using mean-variance analysis. More specifically, he showed that by varying the weights of the respective tax products in the overall tax mix, one could obtain a set of tax product combinations that would provide the highest possible growth rate for a given level of instability of a tax portfolio. Instability of a single tax product was defined as the potential variability or dispersion in the revenue derived from this product, and was approximated using the standard deviation of tax revenue. Growth in an individual tax product was set equal to the regression coefficient on a time variable that was regressed on the tax revenue collections from the tax product, while taking into account the applicable tax rate. The correlation of the individual tax products were taken into consideration using quadratic programming in order to obtain the overall instability of the tax structure.

The analysis by White (1983) failed to consider tax revenue collections in real terms. Since a stable tax structure is one that would produce consistent real income over a specified period, it would make sense to analyse tax revenue on an inflation-adjusted basis. Misiolek & Perdue (1987) addressed this aspect, and compared the inflation-adjusted set of optimal tax structures and the nominal frontier. Their results suggest that the optimal set of tax structures in nominal terms is inefficient when the objective of government is to maintain a level of stability in the provision of public goods and services. Another important observation was that the portfolio optimisation technique can be extremely sensitive to changes in the inputs, and caution should be exercised before recommending changes in the existing tax structure.

Up until now, the focus was on using the growth, variance and covariance characteristics of individual tax products to obtain an efficient set of tax product combinations that would maximise growth at a given level of instability. However, policymakers also want to incorporate other goals when analysing optimal tax structures. Harmon & Mallick (1994) extended the traditional two-goal efficiency frontier analysis consisting of growth and stability to include equity considerations. They introduced the objective of vertical equity to account for tax distributional considerations and estimated a three-goal efficient frontier for New York state taxes. They concluded that when equity considerations were included, the actual tax product portfolio appeared to be fairly efficient, which was not the case when the two-goal efficient frontier was used as the benchmark.

Gentry & Ladd (1994) provided additional refinements and noted that policymakers tend to focus on fairness, efficiency and income predictability when deciding on a certain mix of taxes. Their U.S. based study focused on two states, Massachusetts and North Carolina, and their objective was to contrast these states in terms of revenue growth, stability, equity and competitiveness with other states. By using the portfolio approach to depict the efficient set of tax structures, this approach endeavoured to help policymakers think about the trade-offs among these characteristics in a systematic way. Other interesting observations were that income taxes tend to dominate the optimal set of portfolios over a range of growth rates, while differences in a state's economy and the characteristics of the taxes within that state will inform the ultimate tax structure.

# 3. Analytical framework

The use of modern portfolio theory to derive an optimal tax structure can be seen as supplementary to the well-established field of optimal tax theory. According to Gentry & Ladd (1994: 749), an optimal tax structure is one which maximises some explicit social welfare function while taking into consideration tax-induced inefficiencies and distortions. While standard optimal tax models have been useful in providing theoretical insights, some authors contend that the practical application has been insufficient given their level of abstraction. A large amount of data is required to accurately measure efficiency losses, while the usefulness of the social welfare function in determining fairness will depend on how well the function is specified. In contrast, modern portfolio theory as it applies to tax revenue structures is much less data intensive. No explicit models of economic behaviour or social welfare functions are needed in order to implement the optimisation (although it can be incorporated if desired). Instead, the inputs are mainly based on the historical relationship of the different tax products with the economy. It should be noted that less emphasis is placed on economic efficiency as compared to optimal tax theory, while factors such as the stability of the tax system are given more importance.

## 3.1. Classic modern portfolio theory in the context of tax revenue

For this paper only revenue growth and stability was considered given insufficient data to measure the degree of inequality in each tax category, while tax competition on a state level is not applicable to the South African tax structure. Given the above discussion, the formal analytical framework for analysing the South African tax revenue structure using modern portfolio theory can be outlined as follows:

Assume that a given tax revenue structure or portfolio consists of $N$ different tax categories with each category constituting some percentage of the total tax portfolio. The weighting structure can then be expressed as:

$$\sum_{i=1}^{N} w_i = 1$$

where $w_i$ represents the weight of the $i$-th tax product in the overall tax structure.

Using notation consistent with Fabozzi *et al* (2007: 24-50), the returns of the $N$ individual tax categories can be expressed in vector form as $\boldsymbol{R} = (R_1, R_2, \ldots, R_N)'$ while the expected returns can be expressed as vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)'$. The variance-covariance matrix with N × N dimensions can be expressed as:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{bmatrix}$$

with $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ representing the covariance between tax category $i$ and $j$, $\sigma_{ii}$ representing the variance of tax product $i$, and $\rho_{ij}$ being the correlation between tax category $i$ and $j$. Based on the assumptions above, the return on the overall tax portfolio can be expressed as $R_p = w'\boldsymbol{R}$ and is assumed to be a random variable with an expected return of $\mu_p = w'\boldsymbol{\mu}$ and variance $\sigma_p^2 = w'\boldsymbol{\Sigma}w$.

By varying the weights of the tax products that constitute the tax portfolio, the policy maker can choose among the set of mean-variance combinations. The weights for a particular mean-variance combination can be calculated by choosing a target mean return or growth value, $\mu_0$. Assuming the policy maker's only objective is to minimise tax revenue volatility (or instability) at a given level of return, or stated equivalently, to maximise the tax revenue portfolio growth potential at a given level of portfolio volatility, the constrained minimisation problem can be written as:

$$\min_{w} w'\Sigma w$$

$$\text{subject to } \mu_0 = w'\boldsymbol{\mu}$$

$$\text{and } w'\boldsymbol{\iota} = 1,\ \boldsymbol{\iota}' = [1, 1, \ldots, 1]$$

This quadratic optimisation problem with equality constraints can be solved by the method of Lagrange multipliers, with the Lagrangian expressed as:

$$L = w'\mathbf{\Sigma}w + \delta_1(\mu - w'\boldsymbol{\mu}) + \delta_2(1 - w'\boldsymbol{\iota})$$

Setting the partial derivatives of the Lagrangian equal to zero and solving the equation results in:

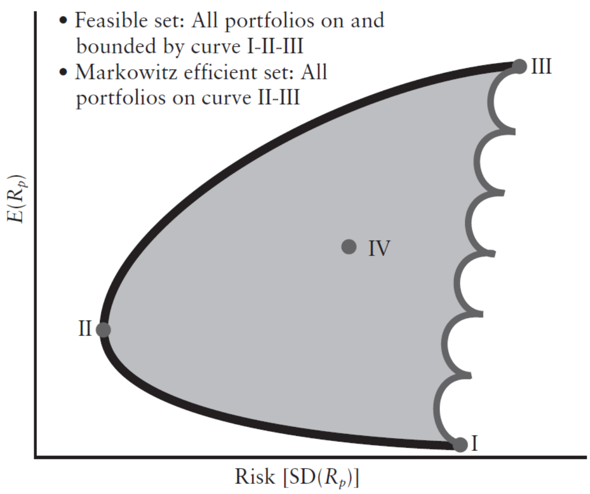
$$w = \mathbf{g} + \mathbf{h}\mu_0$$

$$\text{where } \mathbf{g} = \frac{1}{ac-b^2}\, \mathbf{\Sigma}^{-1}[c\boldsymbol{\iota} - b\boldsymbol{\mu}],\ \mathbf{h} = \frac{1}{ac-b^2}\, \mathbf{\Sigma}^{-1}[a\boldsymbol{\mu} - b\boldsymbol{\iota}]$$

$$\text{and } a = \boldsymbol{\iota}'\mathbf{\Sigma}^{-1}\boldsymbol{\iota},\ b = \boldsymbol{\iota}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu},\ c = \boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$$

The set of portfolios that can be constructed using the mean-variance optimisation framework is called the feasible set. The feasible set covers the grey area in Figure 1. The Markowitz efficient set, also known as the mean-variance efficient portfolios, is bounded by point II and III, and exhibit the lowest level of risk for a given level of return. Portfolios lying between point I and II are inferior to the Markowitz efficient portfolios, since they exhibit more risk for each level of return. Point II is also known as the global minimum variance portfolio as it contains the smallest variance of the set of efficient portfolios.

Figure 1: The efficient frontier




*Source:* Focardi *et al (2007: 475)*

The optimal tax revenue portfolio chosen by the policy maker will depend on the policy maker's risk preference, which is represented by a utility function. Based on the discussion by Fabozzi (2007: 45-50), the chosen utility function will represent a trade-off between computational intensity and accuracy. The quadratic utility function has proven to be a good approximation to more general (and computationally intensive) utility functions.

Moreover, studies have shown that portfolio optimisation is fairly robust to a change in the utility function, assuming absolute risk aversion remains constant. This may explain why the quadratic utility function is the most commonly used function in practice. However, it may be the case that this conclusion only holds for assets with elliptical return distributions. Since it will be assumed in this paper that policy makers only care about the expected growth rate and the variability of the growth rate, the computationally simple case whereby the policy maker is risk averse (represented by a quadratic utility function) will be applicable.

As was shown by Sharpe (1964), Tobin (1958) and Lintner (1965), the efficient set of portfolios obtained through mean-variance analysis without considering a risk-free asset is inferior to the set of portfolios that can be attained by including a risk-free asset. The idea of combining a risk-free asset[4] with the market portfolio to construct minimum variance portfolios that are superior to the original Markowitz efficient frontier would make sense for most investors. Although an interesting concept from a tax revenue portfolio context, it would not make sense from a policy maker's perspective to consider risk-free cash holdings as part of the tax portfolio. As a result, the capital market line, which is the line that runs from the risk-free rate and is tangent to the efficient frontier, will not be derived for the tax revenue portfolio.

### 3.2. Further refinements and non-standard constraints

The original mean-variance framework as described above can be extended to incorporate non-standard features and restrictions in order to make it more applicable to the tax revenue environment. The first constraint from a tax revenue portfolio perspective will involve limiting the tax category weights to non-negative values (i.e. long-only constraint), since the financial market activity known as short selling[5], is not applicable in a tax revenue environment. This restriction can be represented as $w_i \geq 0$.

The second restriction that will be imposed is the so-called holding constraint. This restriction ensures firstly that a particular tax revenue portfolio is not too largely concentrated in one or two tax products. Second, the restriction accounts for the fact that mean variance analysis may lead to unstable portfolio weights or corner solutions (Fabozzi 2007: 391). The restriction involves specifying range limits $L_i$ and $U_i$, which represents

[4] The risk-free rate is the rate of return that an asset with zero risk will produce in theory, since no asset in the real-world has zero risk attributes. The yield on short-dated government bonds are normally used as a proxy for the risk-free rate.
[5] The process of selling assets short involves selling a security that the investor *does not own* (emphasis added) by borrowing the security, with the hope of buying back the security at a lower price in the future.

vectors of the lower and upper limits respectively of each tax category *i*. This constraint can be represented formally as:

$$L_i \leq w_i \leq U_i$$

No guidance is given in the tax portfolio literature as to the most appropriate method of determining the limits. It was eventually decided to calculate the minimum and maximum weights of each tax category over a ten year period as an approximation of the lower and upper limits respectively. Adding these constraints would produce an efficient frontier that would necessarily be inferior to the unconstrained efficient frontier. On the positive side, these constraints have been shown to limit portfolio volatility, increase convergence between the true efficient frontier and the actual frontier, and decrease downside risk (see Fabozzi 1998, Chapter 4).

A further modification to the original framework that will be considered is the portfolio risk measure. In the original mean-variance framework, the variance of returns was used as an approximation for portfolio risk. Based on Markowitz (1959: 188-201), the variance has the unattractive property of treating excessive growth or contraction as equal. He proposed the use of the semi-variance, which only penalises excessive variance on the downside. Similarly, excessive growth of a tax product would be viewed as a favourable outcome, which makes the semi-variance a more appropriate risk measure than the variance of returns. Details on the calculation of the semi-variance will be discussed in the section pertaining to parameter estimation.

## 4. Application to the South African tax revenue portfolio

The modified framework of portfolio optimisation described in the previous section will be referred to as mean-semi-variance optimisation in the rest of the paper. This framework will be applied to the South African tax portfolio, starting with the variable description and the necessary adjustments to the original dataset, followed by the estimation of the parameters and the derivation of the efficient frontier given the specified objective function and constraints. A discussion of the results will follow, along with the implications for policy makers as well the limitations associated with the methodology.

## 4.1. Description of dataset

All tax revenue data was sourced from the SARS Historical Revenue Database. It should be noted at this point that only tax revenue was considered for this study; non-tax revenue instruments was not considered for this study. The relatively large number of individual tax products was aggregated into eight categories in order to minimise the number of parameters to be estimated. The eight categories were:

| Variable short name | Variable description |
|---|---|
| **PIT** | Personal income tax (excluding interest on overdue tax) |
| **CIT** | Corporate income tax (excluding interest on overdue tax) |
| **DWT** | Dividend withholding tax (preceded by the secondary tax on companies) |
| **VAT** | Value-added tax (net of refunds) |
| **FUEL** | Fuel levy (net of refunds) |
| **EXCISE** | Total excise duties (consisting of ad-valorem and specific excise duties) |
| **CUSTOMS** | Customs duties |
| **OTHER** | Other taxes (including interest on overdue income tax) |

Annual data for the period 1994/95 to 2012/13 was used firstly to take into consideration the changeover to a democratic political system. Second, although resulting in a limited number of observations, the annual frequency was chosen to avoid the challenges associated with seasonality as well as to take advantage of tax proposal data that are released annually by the National Treasury.

Before one can estimate the inputs necessary to perform mean-semi-variance analysis, one needs to make adjustments to the original revenue dataset. First, discretionary changes[6] needs to be taken into consideration in order to ensure comparability over the measured period; if not taken into account, the growth values obtained will provide a distorted picture of the actual growth potential of the tax product. The proportional adjustment method as originally proposed by Prest (1962) will be used to adjust for the effect of discretionary changes. The process essentially involves compiling actual revenue collections for each of the eight tax categories as defined earlier, constructing a series containing the discretionary changes for each of the eight tax categories and adjusting the actual revenue series for these discretionary changes using some pre-set formula (shown below).

[6] Discretionary changes, according to Sen (2009), are legislative changes in the form of tax rate or tax base amendments, the introduction of new taxes as well as specific changes in tax effort.

Data on discretionary changes was obtained from annual Budget Review publication of the National Treasury. The reference year was chosen as 2012/13, which means the revenue values in the prior years will be adjusted to reflect the tax structure that prevailed in 2012/13. The adjusted tax series for 2012/13 will therefore be equal to the actual or unadjusted revenue series:

$$AT_t = T_t$$

where $AT_t$r represents the adjusted tax series in the current or reference year (i.e. year *t*) while $T_t$ represents the actual tax series in the reference year. For years prior to the 2012/13 reference year, the adjusted tax series will be calculated using the following expression:

$$\mathrm{AT}_j = \mathrm{T}_j \prod_{k=j+1}^{t} \left[\frac{\mathrm{T}_k}{T_k - D_k}\right] \text{ where } \mathrm{j} < \mathrm{t}$$

where $\mathrm{AT}_j$ represents the adjusted tax revenue series for years prior to *t*, $\mathrm{T}_j$ represents the original tax revenue series for years prior to *t*, while $D_k$ represents the discretionary measures in the *k*-th period. Note that the proportional impact of current period discretionary changes on prior periods is also accounted for.

As noted by Wolswijk (2007: 13), the usefulness of this method will depend on the availability and quality of the revenue forgone or gained due to discretionary changes. The fact that these estimates are done by the National Treasury on an ex-ante basis will further contribute to estimation error. Table 1a and Table 1b give an indication as to how discretionary changes have affected the selected tax categories over time. The indicated tax categories are typically affected by discretionary changes on a frequent basis.

Table 1a: Adjustment for discretionary changes to selected tax categories, 2002/03 – 2012/13

| R million | **PIT** | | **CIT** | | **FUEL** | | **EXCISE** | |
|---|---|---|---|---|---|---|---|---|
| | **Original** | **Adjusted** | **Original** | **Adjusted** | **Original** | **Adjusted** | **Original** | **Adjusted** |
| **2002/03** | 94 337 | 54 506 | 55 745 | 48 665 | 15 334 | 30 758 | 11 473 | 23 857 |
| **2003/04** | 98 495 | 64 666 | 60 881 | 54 947 | 16 652 | 32 116 | 12 381 | 25 554 |
| **2004/05** | 110 982 | 75 531 | 70 782 | 63 883 | 19 190 | 35 257 | 14 082 | 26 541 |
| **2005/06** | 125 645 | 90 349 | 86 161 | 81 150 | 20 507 | 35 930 | 15 704 | 27 150 |
| **2006/07** | 140 578 | 109 806 | 118 999 | 112 454 | 21 845 | 38 275 | 17 652 | 28 187 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **2007/08** | 168 774 | 138 759 | 140 120 | 133 156 | 23 741 | 39 932 | 19 699 | 29 228 |
| **2008/09** | 195 146 | 166 771 | 165 539 | 164 344 | 24 884 | 39 752 | 21 354 | 29 681 |
| **2009/10** | 205 145 | 186 896 | 134 883 | 134 902 | 28 833 | 38 249 | 22 565 | 28 445 |
| **2010/11** | 226 925 | 211 658 | 132 902 | 134 271 | 34 418 | 40 882 | 24 564 | 28 129 |
| **2011/12** | 250 400 | 241 808 | 151 627 | 152 683 | 36 602 | 41 220 | 27 239 | 28 977 |
| **2012/13** | 275 805 | 275 805 | 158 947 | 158 947 | 40 320 | 40 320 | 30 691 | 30 691 |

It can be deduced from Table 1a that for tax categories exposed to discretionary changes that result in foregone revenue, which is mostly the case for tax proposals relating to income tax, the adjusted data series would be less than the original data series. This is especially evident in the case of personal income tax. During 2002/03 and 2003/04, tax relief derived from adjustments made to the personal income tax rate structure amounted to R15 billion and R13.3 billion respectively. Other notable periods of tax relief was during 2006/07 and 2009/10 when relief solely attributed to changes in the tax rate structure amounted to R13.5 billion and R13.0 billion respectively. Discretionary changes have been less dramatic in terms of the corporate income tax; 2005/06 and 2008/09 received comparatively large tax relief of R2.0 billion and R5.0 billion which can be directly attributed to a reduction in the corporate tax rate. When comparing the growth rates in for income tax collections before and after proposals, it is quite evident that discretionary changes play a significant role. When tax relief is stripped out, growth in income tax collections look significantly higher especially during years in which the amount of tax relief granted was significant.

Table 1b: Adjustment for discretionary changes to selected tax categories, 2002/03 – 2012/13

| %, nominal | **PIT** | | **CIT** | | **FUEL** | | **EXCISE** | |
|---|---|---|---|---|---|---|---|---|
| | **Original** | **Adjusted** | **Original** | **Adjusted** | **Original** | **Adjusted** | **Original** | **Adjusted** |
| **2002/03** | 4.3 | 19.1 | 27.5 | 28.1 | 2.7 | 2.7 | 8.2 | 2.2 |
| **2003/04** | 4.3 | 17.1 | 8.8 | 12.1 | 8.2 | 4.3 | 7.6 | 6.9 |
| **2004/05** | 11.9 | 15.5 | 15.1 | 15.1 | 14.2 | 9.3 | 12.9 | 3.8 |
| **2005/06** | 12.4 | 17.9 | 19.7 | 23.9 | 6.6 | 1.9 | 10.9 | 2.3 |
| **2006/07** | 11.2 | 19.5 | 32.3 | 32.6 | 6.3 | 6.3 | 11.7 | 3.8 |
| **2007/08** | 18.3 | 23.4 | 16.3 | 16.9 | 8.3 | 4.2 | 11.0 | 3.6 |
| **2008/09** | 14.5 | 18.4 | 16.7 | 21.0 | 4.7 | -0.5 | 8.1 | 1.5 |
| **2009/10** | 5.0 | 11.4 | -20.5 | -19.7 | 14.7 | -3.9 | 5.5 | -4.3 |
| **2010/11** | 10.1 | 12.4 | -1.5 | -0.5 | 17.7 | 6.7 | 8.5 | -1.1 |
| **2011/12** | 9.8 | 13.3 | 13.2 | 12.9 | 6.2 | 0.8 | 10.3 | 3.0 |
| **2012/13** | 9.7 | 13.2 | 4.7 | 4.0 | 9.7 | -2.2 | 11.9 | 5.7 |

In contrast to the income tax category, increases in the fuel levy rate and excise duties on certain products have contributed strongly to the growth in tax collections observed over the measured period. In the case of net fuel levy collections, the amount of revenue that was attributed solely to discretionary changes amounted to more than R16 billion cumulatively over the past five financial years. When these changes are accounted for, the growth in net fuel levy collections is dramatically reduced. Similarly, although in a less dramatic fashion, the growth in total excise duty collections over the measured period are significantly less when discretionary changes are taken into consideration.

Following the adjustment for discretionary changes, the data needs to be adjusted for inflation. It is recognised that the consumer price index will not be representative of price changes in all eight tax categories. The consumer price index may provide a fairly accurate approximation of inflation in terms of personal income tax given that wage increases are normally based on some future estimate of the consumer price index. However, in the case of corporate income tax it is unlikely to provide a good approximation of inflation, given the heterogeneous population of companies registered for corporate income tax as well as the uncertainty involved in measuring the degree to which companies can pass through price changes to their client base. However, no reference to alternative price deflators is made in the literature. Future research efforts will focus on refining this aspect of the methodology. Table 2 provides figures for selected tax categories in terms of nominal and inflation-adjusted figures.

Table 2: Inflation-adjusted percentage change for selected tax categories, 2002/03 – 2012/13

| %, real | **PIT** | | **CIT** | | **VAT** | | **CUSTOMS** | |
|---|---|---|---|---|---|---|---|---|
| | **Nominal** | **Real** | **Nominal** | **Real** | **Nominal** | **Real** | **Nominal** | **Real** |
| **2002/03** | 19.1 | 9.2 | 28.1 | 18.2 | 13.9 | 4.0 | 7.8 | -2.1 |
| **2003/04** | 17.1 | 13.8 | 12.1 | 8.8 | 14.0 | 10.7 | -9.6 | -12.9 |
| **2004/05** | 15.5 | 13.5 | 15.1 | 13.1 | 19.6 | 17.6 | 41.9 | 39.9 |
| **2005/06** | 17.9 | 14.4 | 23.9 | 20.4 | 15.3 | 11.8 | 35.1 | 31.6 |
| **2006/07** | 19.5 | 14.4 | 32.6 | 27.5 | 16.9 | 11.8 | 25.8 | 20.7 |
| **2007/08** | 23.4 | 15.6 | 16.9 | 9.1 | 11.2 | 3.5 | 11.1 | 3.3 |
| **2008/09** | 18.4 | 7.7 | 21.0 | 10.3 | 2.6 | -8.2 | -15.1 | -25.9 |
| **2009/10** | 11.4 | 5.8 | -19.7 | -25.3 | -4.2 | -9.8 | -15.0 | -20.6 |
| **2010/11** | 12.4 | 8.7 | -0.5 | -4.2 | 21.6 | 17.8 | 30.8 | 27.1 |
| **2011/12** | 13.3 | 7.9 | 12.9 | 7.4 | 4.0 | -1.4 | 25.0 | 19.6 |
| **2012/13** | 13.2 | 7.8 | 4.0 | -1.4 | 12.2 | 6.8 | 10.9 | 5.5 |

Based on the output from Table 2, assuming the consumer price index is representative of price changes in all eight tax categories, it can be observed that real growth in personal income tax collections have remained in the mid to high single digit level since the start of the recession. This was preceded by a five-year period of significant growth, which was primarily the result of base broadening, robust employment gains and real income growth. It can also be deduced that corporate income tax collections have been very sensitive to movements in the business cycle. As evidence, between 2000/01 and 2007/08 real annual growth in corporate income tax collections averaged 19.2 percent, supported by strong economic growth and company profits (based on the performance of the gross operating surplus). However, three out of the last five financial years experienced negative inflation-adjusted growth in company tax collections as a result of the global financial crisis. Real growth in value-added tax collections was also significantly affected by the slowdown in real household consumption during 2008/09 and 2009/10, but recovered markedly the following year. Growth in real customs duty collections was affected significantly by the drastic fall in international trade activity following the start of the global financial crisis in late 2007. The subsequent years of 2008/09 and 2009/10 saw double-digit rates of contraction as a result of negative growth in the value of imports.

## 4.2. Parameter estimation

Following the adjustments to the original dataset, the next step is to estimate the inputs necessary to perform mean-semi-variance optimisation. According to Fabozzi (2007: 211), estimates for expected growth has roughly ten times more importance in optimisation as compared to the covariance matrix, while errors in the variances are approximately twice as important as those in the co-variances. It would therefore be crucial to obtain accurate estimates of expected growth for each tax revenue product, followed by the variances.

There are various techniques for estimating expected growth and co-variances, ranging from using historical data as a proxy for future values to multi-factor models and general equilibrium theories. Historical data have proven to be a poor predictor of future returns given changes in general economic conditions, political considerations as well as the business cycle. As a consequence, it has become common investment practice to use some factor model to generate forward-looking return estimates, the Black-Litterman model or shrinkage estimators (Fabozzi 2007: 146-149). Estimating multi-factor models for each tax product or using the Black-Litterman model would be a research topic on its own and, given the focus of this paper, will not be considered. Shrinkage estimators will therefore be used to impose more structure on the sample estimators.

Tsay (2002: 2-6) notes that most financial studies make use of data in differenced form (i.e. growth values) as opposed to levels, given its more attractive statistical properties and scale-free interpretation. In addition, continuously compounded (or logarithmic) growth values are normally used as opposed to simple arithmetic growth values. The decision to use logarithmic returns was based on its more tractable statistical properties and the easy extension of one-period returns into a multi-period return.

Based on the above discussion, the natural logarithm of simple growth in a tax product over one period can be expressed as:

$$r_t = \ln(1 + R_t) = \ln \frac{T_t}{T_{t-1}}$$

where $r_t$ represents continuously compounded growth, $R_t$ represents simple arithmetic growth, while $T_t$ and $T_{t-1}$ represents revenue collected through some tax instrument $T$ in the current and previous period respectively. To obtain the compound average growth rate for a tax instrument over $k$ years, the average growth rate can be calculated using the following expression:

$$r_t[k] = \frac{1}{k} \sum_{j=0}^{k-1} \ln(1 + R_{t-j})$$

Building on this foundation, the tax portfolio's continuously compounded growth in year *t* can be approximated using the expression:

$$r_{p,t} \approx \sum_{i=1}^{N} w_i r_{i,t}$$

where $r_{p,t}$ represents the continuously compounded growth value of the portfolio at time $t$, $w_i$ is the weight attached to tax category *i* and $r_{i,t}$ represents the return on tax category *i* at time *t*. The compound average growth rate for the tax portfolio over *k* years can be calculated in a similar fashion as was done for the individual tax instruments. In this paper, the value of *k* will be equal to eighteen since the period of measurement is from 1994/95 to 2012/13.

To address some of the shortcomings inherent in historical data, the shrinkage estimator as proposed by Jorion (1986) will be used to adjust the expected growth values for each tax category. The Jorion shrinkage estimation procedure involves “shrinking” the sample mean growth values toward a common value, with the aim of reducing estimation error when more than two assets are present in a portfolio. This method follows from the result that the sample mean is an “inadmissible estimator” when more than two parameters are considered, based on work done by Stein (1955). The so-called James-Stein shrinkage estimator, which was proven to be a superior estimate of the population mean as compared to the sample mean, can be expressed as follows, based on Fabozzi (2007: 215-217):

$$\hat{\boldsymbol{\mu}}_{JS} = (1 - w)\hat{\boldsymbol{\mu}} + w\mu_0\boldsymbol{\iota}$$

where $\hat{\boldsymbol{\mu}}$ represents the sample mean, $w$ represents the shrinkage intensity and vector $\mu_0\boldsymbol{\iota}$ represents the shrinkage target. The Jorion shrinkage estimator, which builds on the James-Stein shrinkage estimator, specifies the shrinkage target $\mu_g\boldsymbol{\iota}$ and shrinkage intensity $w$ as follows:

$$\mu_{\mathrm{g}} = \frac{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\hat{\boldsymbol{\mu}}$$

$$\text{and } w = \frac{N + 2}{N + 2 + T\left(\hat{\boldsymbol{\mu}} - \mu_{\mathrm{g}}\boldsymbol{\iota}\right)'\boldsymbol{\Sigma}^{-1}\left(\hat{\boldsymbol{\mu}} - \mu_{\mathrm{g}}\boldsymbol{\iota}\right)}$$

where $\mu_g$ represents the return on the minimum variance portfolio. Since $\boldsymbol{\Sigma}$ represents the population covariance matrix, it is approximated in practice using:

$$\hat{\boldsymbol{\Sigma}} = \frac{T - 1}{T - N - 3}\boldsymbol{S}$$

Based on the discussion by Fabozzi (2007: 217), using the shrinkage estimator as discussed earlier may lead to less variability in the portfolio weights and improved out-of-sample performance. Table 3 compares the sample mean returns with the mean growth values that has been adjusted using the Jorion shrinkage estimator.

Table 3: Adjusted and unadjusted mean growth estimates for the main tax categories

| %, real | PIT | CIT | DWT | VAT | FUEL | EXCISE | CUSTOMS | OTHER |
|---|---|---|---|---|---|---|---|---|
| **Sample mean growth (unadjusted)** | 9.6 | 7.4 | 8.6 | 4.8 | -3.4 | -4.1 | 6.4 | 4.5 |
| **Sample mean growth (adjusted)** | 9.5 | 7.3 | 8.6 | 4.8 | -3.2 | -3.9 | 6.4 | 4.6 |

The second parameter to be estimated is the semi-variance. This downside measure, first suggested by Markowitz (1959), serves as an alternative to the variance as a proxy for risk in a portfolio. Jin *et al* (2006) was the first to establish the actual existence of a semi-variance efficient frontier and also showed that mean-semi-variance efficient strategies are attainable regardless of market conditions or a security's return distribution.

One potential challenge pertaining to using the semi-variance in portfolio optimisation, according to Estrada (2008: 58), is that it does not have a closed-form solution. This is because the so-called semi-covariance matrix has proven to be endogenous. It therefore requires some form of numerical algorithm in order to solve it. Estrada (2008) shows that mean-semi-variance problems can be solved using the familiar closed-form solution similar to mean-variance portfolios when some approximation or heuristic is used. First, using the approach of Estrada (2008), the semi-covariance between tax categories *i* and *j* relative to a benchmark *B* can be written as:

$$\Sigma_{ijB} = \frac{1}{T}\sum_{t=1}^{T}[\text{Min}(r_{it} - B, 0).\,\text{Min}(r_{jt} - B, 0)]$$

where $r_{it}$ and $r_{jt}$ represents the continuously compounded growth rate in tax category *i* and *j* at time *t* respectively. The obtained matrix has the properties of being both symmetric and exogenous. The value of the benchmark *B* can take on any value; the guidance given by Estrada (2008: 68-70) is to set this value equal to zero. Second, in order to obtain the semi-variance of the tax portfolio relative to a benchmark or hurdle rate *B* without resorting to an algorithm, the following approximation or heuristic can be used:

$$\Sigma_{pB}^2 \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \Sigma_{ijB}$$

where $w_i$ and $w_j$ represent the proportions of the tax portfolio constituted by tax category *i* and *j* respectively. This heuristic approach has proven to be quite accurate and requires much less computational power (Estrada

2008: 63-67). Table 4 shows the semi standard deviation as well as the standard deviation of each tax category. Note that the benchmark value of 6 percent is an arbitrary number and was chosen based on the fact that the suggested benchmark value of 0 percent results in the semi standard deviation of real personal income tax collections to be equal to zero, which creates estimation problems.

Table 4: Comparison of different risk measures

| Risk measure (%) | PIT | CIT | DWT | VAT | FUEL | EXCISE | CUSTOMS | OTHER |
|---|---|---|---|---|---|---|---|---|
| Standard deviation | 3.2 | 13.8 | 24.1 | 7.2 | 4.8 | 4.9 | 17.9 | 13.4 |
| Semi standard deviation (benchmark = 6%) | 0.2 | 9.1 | 14.3 | 5.7 | 10.5 | 11.2 | 12.4 | 10.6 |

Comparing the standard deviation across the different tax categories, it can be seen that growth in real dividend withholding tax, corporate income tax and customs duty collections are the most volatile tax categories. The least volatile tax categories are personal income tax, the fuel levy and total excise duty collections. However, based on the semi standard deviation measure relative to a uniform benchmark of 6 percent, real fuel levy and total excise duty collections exhibit significant downside volatility, which is not evident in the standard deviation. Personal income tax collections remain the tax category with the lowest risk even when focusing exclusively on below-target deviations, followed by real value-added tax and corporate income tax. To the author's knowledge, no other studies focusing on tax portfolio optimisation have made use of the semi standard deviation; the obtained results are therefore not comparable to the existing literature.

Table 5: Risk-return trade-off in each tax category

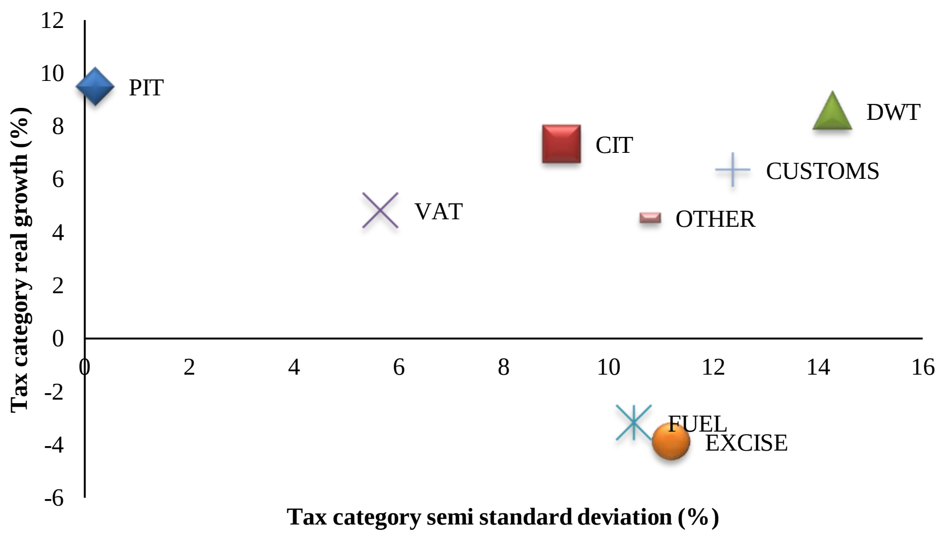


Table 5 provides a visual inspection of the trade-off in growth versus variability for each of the eight tax categories. Low-volatility portfolios as well as high-return portfolios would likely contain a large proportion of personal income tax due to its favourable risk-return profile. Personal income tax can be combined with the value-added tax and corporate income tax to form a tax portfolio with lower risk per unit of return as compared to the risk-return profile of the individual tax categories, the extent of which will depend on the co-movement between these tax categories. It can be observed that customs duty collections are inferior to corporate income tax collections in terms of the growth-variability trade-off, given that corporate income tax has a higher real growth rate and lower downside risk. The dividend withholding tax will only feature strongly in portfolios with a high target return, all other factors remaining constant. Fuel levy and total excise duty collections are not expected to play a significant role in an optimal portfolio given their weak real growth and high downside volatility; this effect may be slightly offset by the possible diversification benefits these categories may provide.

The third and final parameter to be estimated is the correlation matrix. As was done in the case of the mean growth values, the logic behind shrinkage estimators can also be extended to the covariance matrix (and by extension the correlation matrix), based on the paper by Ledoit & Wolf (2004). According to Ledoit & Wolf (2004: 110-111), the estimation error present in the sample covariance is significant, especially in situations where the number of observations is low relative to the number of investment securities considered. The inherent estimation error is normally upward biased in the case of extremely high coefficients, and should be "pulled down" to compensate for this upward bias. Similarly, the downward bias in extremely negative estimates should be "pulled up". These corrections can be done using a shrinkage estimator for the covariance matrix. This logic also applies in the case of tax revenue portfolios. The shrinkage target will be represented by the constant-correlation model. Based on the constant-correlation model, the average of all the sample correlations serves as the estimator of the common constant correlation, which along with the vector of sample variances denotes the shrinkage target. Using the more intuitive notation of Fabozzi (2007: 217-220), the Ledoit-Wolf shrinkage estimator can be written as:

$$\hat{\mathbf{\Sigma}}_{LW} = w\hat{\mathbf{\Sigma}}_{CC} + (1-w)\hat{\mathbf{\Sigma}}$$

where $\hat{\mathbf{\Sigma}}$ represents the sample covariance matrix, $\hat{\mathbf{\Sigma}}_{CC}$ represents the sample covariance matrix with constant correlation. The optimal weighting structure, *w*, is one which minimises the expected distance between the shrinkage estimator and the true covariance matrix (Ledoit & Wolf 2004: 113). The formula for determining the optimal weighting structure will not be elaborated upon in this paper (see Ledoit & Wolf (2004), Appendix B). The adjusted correlation matrix using the semi-covariance and semi standard deviation is displayed below.

Table 6: Correlation matrix using the semi-covariance and downside deviation of the respective tax categories

| | PIT | CIT | DWT | VAT | FUEL | EXCISE | CUSTOMS | OTHER |
|---|---|---|---|---|---|---|---|---|
| **PIT** | 1.000 | 0.997 | 0.992 | 0.999 | 1.000 | 1.000 | 0.995 | 0.997 |
| **CIT** | 0.997 | 1.000 | 0.992 | 0.997 | 0.997 | 0.997 | 0.995 | 0.995 |
| **DWT** | 0.992 | 0.992 | 1.000 | 0.992 | 0.991 | 0.991 | 0.991 | 0.992 |
| **VAT** | 0.999 | 0.997 | 0.992 | 1.000 | 1.000 | 0.999 | 0.997 | 0.998 |
| **FUEL** | 1.000 | 0.997 | 0.991 | 1.000 | 1.000 | 1.000 | 0.996 | 0.998 |
| **EXCISE** | 1.000 | 0.997 | 0.991 | 0.999 | 1.000 | 1.000 | 0.995 | 0.997 |
| **CUSTOMS** | 0.995 | 0.995 | 0.991 | 0.997 | 0.996 | 0.995 | 1.000 | 0.997 |
| **OTHER** | 0.997 | 0.995 | 0.992 | 0.998 | 0.998 | 0.997 | 0.997 | 1.000 |

The correlation matrix pertaining to the semi-variance and semi-covariance of the eight tax categories shows the high degree of co-movement in the downside volatility of each tax category. This result essentially points out the fact that the volatility of the different tax categories below the stated threshold of 6 percent is highly correlated, and does not provide much scope for diversification during difficult times. The correlation matrix using the covariance and the standard deviation is included for purposes of comparison in Table 7. Note that the optimal set of portfolios derived using the semi standard deviation as risk measure can look significantly different from the optimal set obtained by using the standard deviation.

Table 7: Correlation matrix using the covariance and standard deviation of the respective tax categories

| | PIT | CIT | DWT | VAT | FUEL | EXCISE | CUSTOMS | OTHER |
|---|---|---|---|---|---|---|---|---|
| **PIT** | 1.000 | 0.306 | 0.462 | 0.515 | 0.590 | 0.399 | 0.309 | 0.283 |
| **CIT** | 0.306 | 1.000 | 0.431 | 0.397 | 0.158 | 0.318 | 0.346 | 0.068 |
| **DWT** | 0.462 | 0.431 | 1.000 | 0.350 | 0.283 | 0.139 | 0.411 | 0.426 |
| **VAT** | 0.515 | 0.397 | 0.350 | 1.000 | 0.784 | 0.585 | 0.720 | 0.431 |
| **FUEL** | 0.590 | 0.158 | 0.283 | 0.784 | 1.000 | 0.645 | 0.572 | 0.388 |
| **EXCISE** | 0.399 | 0.318 | 0.139 | 0.585 | 0.645 | 1.000 | 0.315 | -0.016 |
| **CUSTOMS** | 0.309 | 0.346 | 0.411 | 0.720 | 0.572 | 0.315 | 1.000 | 0.612 |
| **OTHER** | 0.283 | 0.068 | 0.426 | 0.431 | 0.388 | -0.016 | 0.612 | 1.000 |

## 4.3. Derivation of the tax portfolio efficient frontier

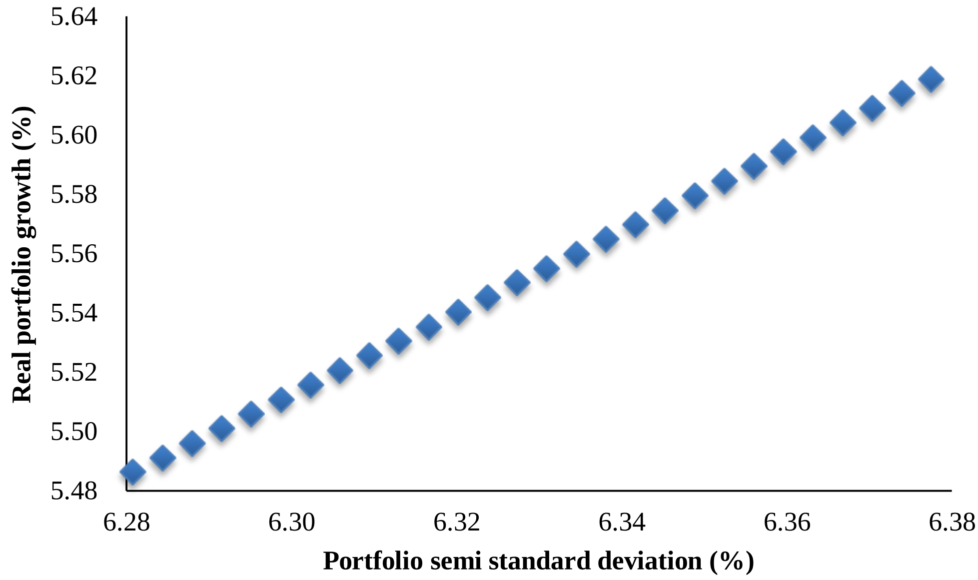
Using the inputs that were calculated in the previous section, the efficient set of tax category combinations can now be derived using the following objective function:

$$\min_{w} \ \Sigma_{pB} \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \Sigma_{ijB}$$

subject to $\mu_0 = w'\boldsymbol{\mu}$, $w'\boldsymbol{\iota} = 1$, and $L_i \leq w_i \leq U_i$ where $L_i$, $U_i \geq 0$

All the variables are defined as discussed earlier. Since the heuristic approach was used to obtain the semi-covariance matrix, standard portfolio optimisation procedures will suffice to obtain the optimal portfolio weights. Many software packages are available to perform portfolio optimisation, which includes R, Matlab, SAS and Microsoft Excel. To facilitate some of the statistical adjustments to the dataset, an Excel based portfolio optimisation software called Hoadley Trading & Investment Tools was used. The expected growth values were estimated using the continuously compounded average return of each tax category over the sample period from 1994/95 to 2012/13. The semi-variance and semi-covariance values were calculated over the sample period with respect to a benchmark return of 6 percent, the value of which was chosen arbitrarily based on the zero semi-variance characteristic of personal income tax relative to the suggested benchmark.

Figure 8: Efficient frontier of real tax portfolio structure using semi standard deviation



At first glance, the efficient frontier of the real tax portfolio as depicted in Figure 8 does not exhibit the curved pattern that is often observed. This can be directly attributed to the high degree of correlation in the downside deviation of the respective tax categories, which leaves little room for downside risk diversification. Generally speaking, the lower the correlation between assets in a portfolio, the larger the diversification benefit observed. Table 9 compares the actual weights of each tax category as a proportion of total real tax collections in 2012/13 along with five different sets of portfolios that are considered Markowitz efficient. The minimum and maximum weights imposed on the tax categories (as discussed earlier) was included to provide context to the results. Five efficient portfolios were chosen (represented by A-E), with A representing the minimum variance portfolio and E the maximum return portfolio. Remember that one efficient portfolio cannot be preferred over the other unless the utility function of the policy maker is known.

Table 9: Two-goal efficient frontier in real terms of the South African tax portfolio

| Tax category | Weight (2012/13) | Min weight | Max weight | Efficient frontier portfolios | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | A | B | C | D | E |
| PIT | 33.89 | 22.41 | 33.89 | 33.89 | 33.89 | 33.89 | 33.89 | 33.89 |
| CIT | 19.53 | 18.95 | 26.42 | 19.55 | 20.55 | 21.55 | 23.54 | 22.12 |
| DWT | 2.43 | 1.77 | 3.30 | 1.77 | 1.77 | 1.77 | 1.77 | 3.30 |
| VAT | 26.52 | 24.80 | 28.90 | 28.90 | 27.90 | 26.90 | 24.91 | 24.80 |
| FUEL | 4.95 | 4.95 | 11.17 | 4.95 | 4.95 | 4.95 | 4.95 | 4.95 |
| EXCISE | 3.77 | 3.77 | 8.89 | 3.77 | 3.77 | 3.77 | 3.77 | 3.77 |
| CUSTOMS | 4.69 | 2.95 | 4.86 | 2.95 | 2.95 | 2.95 | 2.95 | 2.95 |
| OTHER | 4.22 | 4.22 | 5.79 | 4.22 | 4.22 | 4.22 | 4.22 | 4.22 |
| Total | 100.0 | - | - | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |
| Portfolio characteristics: | | | | | | | | |
| Growth | 6.33 | - | - | 6.28 | 6.31 | 6.33 | 6.38 | 6.40 |
| Semi standard deviation | 5.66 | - | - | 5.49 | 5.52 | 5.55 | 5.62 | 5.71 |

Given the favourable growth-variability characteristics of personal income tax collections, its weight in the set of optimal portfolios A-E is equal to its maximum weight as a proportion of real total tax revenue over the past 10 years, which coincidently was the weighting that prevailed during the last financial year. By implication, had this so-called holding constraint not been imposed, the optimal weighting allocated to personal income tax collections would have been much higher. A similar but opposite situation exists for fuel levy, total excise, customs duty and other collections in that the optimal weight allocated to each of these categories are equal to its imposed minimum weight. This is unsurprising given their inferior growth-variability characteristics. The value-added tax features more prominently in low-growth portfolios, and is allocated an increasingly smaller weighting as the required rate of real growth increases. The corporate income tax and the dividend withholding tax features more prominently in high-growth portfolios given their volatile but high real growth potential.

Another important observation is that the current tax portfolio is operating slightly below the efficient frontier. This can be deduced from the real growth rate of 6.33 percent that was achieved with downside risk of 5.66 percent during the 2012/13 financial year. Had the optimal weighting structure prevailed, the same amount of real growth could have been achieved with downside risk of only 5.55 percent.

### 4.4. Discussion of results and limitations

It should be noted at this point that the efficient set of tax portfolios derived in the previous section illustrates the different combinations of growth and downside volatility that policy makers can choose from. However, without knowledge of a policy maker's preferences in terms of these characteristics, one efficient combination of tax categories cannot be preferred over another efficient combination. As noted by Gentry & Ladd (1994: 760-761), it is likely that the real level of tax portfolio growth desired by policy makers will roughly correspond to some "desired" rate of growth in expenditure. As was the case in Gentry & Ladd (1994), income taxes tend to dominate the optimal tax portfolio over a wide range of growth rates. Another interesting outcome was the weak growth-variability characteristics of the fuel levy and total excise duty collections when taking into account discretionary changes and inflation. The effect of revenue diversification in the tax portfolio is evident in that the real growth rate of the portfolio is achieved at a significantly lower level of downside risk as compared to the individual tax categories. However, scope for further diversification by changing the weights of the individual tax categories are limited, given the high degree of co-movement in the downside volatility measures.

One of the deficiencies attached to the approach taken in this paper is the lack of optimisation constraints pertaining to factors other than growth and variability that is considered by policy makers, such as equity and efficiency considerations. In addition, there may be factors other than equity and efficiency that influence the decision-making process of policy makers, but it may be impossible to incorporate it into the optimisation framework. A second limitation relates to endogeneity. Although the expected growth and volatility of a listed company can be taken as exogenous by an investor, policy makers is likely to change the growth and volatility characteristics of a tax product in an attempt to change its relative contribution to the overall tax portfolio. Based on Gentry & Ladd (1994: 761), the tax portfolio model therefore works well when assuming that policy makers cannot make significant adjustments to the overall tax portfolio structure. Third, as was discussed to some extent in the section pertaining to parameter estimation, using historical estimates of growth and volatility as an approximation for expected growth and instability may not be optimal in the presence of structural changes in the economy or significant changes in tax legislation. This limitation also applies to investment portfolios. Finally, it was pointed out to the author of this paper that changing the relative stability of the tax portfolio may distort the role of tax revenue as an automatic stabiliser. Given the limitations discussed above, it would be prudent to exercise caution when interpreting the results of the portfolio optimisation; nevertheless, the portfolio approach does provide an intuitive and practical framework to assist in policy decision making.

## 5. Conclusion

This paper represents the first South African application of modern portfolio theory to derive a set of tax category combinations that would minimise the risk of under-collecting (relative to a specified threshold) at a given level of real growth. The methodology employed also tries to improve on the existing tax portfolio literature by incorporating an alternative measure of risk that focuses exclusively on downside risk which does not penalise tax categories that exhibits a positively skewed growth distribution.

Using the mean, variance and covariance of the specified tax categories, it has been shown that the South African tax portfolio in its current state lies below the efficient frontier curve, indicating that more revenue volatility or risk is currently present than is warranted by the expected real level of growth, and therefore carries unnecessary risk of under-collection. Shrinkage estimators were used to partly account for the weak predictive power of historical estimates of growth and variability.

The so-called tax portfolio model can serve as an additional tool for policy makers in order to think about the trade-offs inherent in changing tax legislation or adding/removing tax instruments from a growth-variability perspective. Optimal taxation theory and the associated models remains the primary source of information when designing a tax system, but due to limitations in terms of practical implementation it should be supplemented with additional methods. The main limitations to the tax portfolio approach in its current form are the lack of equity and efficiency considerations, and the fact that the growth and variability estimates are endogenous. Future research will attempt to address the first limitation, while the issue of endogeneity can to some extent be minimised by imposing holding constraints on the individual tax categories in order to limit the degree to which policy makers can amend the current tax structure composition.

## 6. References


Estrada, J. “Mean-Semivariance Optimization: A Heuristic Approach.” *Journal of Applied Finance* 18 (2008), 57-72.

Fabozzi, F. *Handbook of Portfolio Management.* New Jersey: John Wiley and Sons, 1998.

Fabozzi, F.J., Kolm, P.N., Pachamanova, D.A., and Focardi, S.M. *Robust Portfolio Optimization and Management*. Hoboken, New Jersey: John Wiley & Sons, 2007.

Gentry, W.M., and Ladd, H.F. “State Tax Structure and Multiple Policy Objectives.” *National Tax Journal* 47 (1994), 747-772.

Groves, H.M., and Kahn, C.H. “The Stability Of State And Local Tax Yields.”*American Economic Review* 42 (1952), 87-102.

Harmon, O.R., and Mallick, R. “The Optimal State Tax Portfolio Model: An Extension.” *National Tax Journal* 47 (1994), 395-401.

Jenkins, G.P., Kuo, C., and Shukla, G.P. *Tax Analysis and Revenue Forecasting: Issues and Techniques*. Harvard Institute for International Development, 2000.

Jin, H., Markowitz, H., and Zhou, X.Y. “A Note on Semi-variance.” *Mathematical Finance* 16 (2006), 53-61.

Jorion, P. “Bayes-Stein Estimation for Portfolio Analysis.” *Journal of Financial and Quantitative Analysis* 21 (1986), 279-292.

Ledoit, O., and Wolf, M. “Honey, I Shrunk the Sample Covariance Matrix.” *Journal of Portfolio Management* 30 (2004), 110-119.

Lintner, J. “The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolio and Capital Budgets.” *Review of Economics and Statistics* 47 (1965), 13-37.

Markowitz, H. “Portfolio Selection.” *The Journal of Finance* 7 (1952), 77-91.

Markowitz, H., *Portfolio Selection: Efficient Diversification of Investments*. John Wiley and Sons Publication, 1959.

Misiolek, W.S., and Perdue, D.G. “The Portfolio Approach to State and Local Tax Structures.” *National Tax Journal* 40 (1987).

Prest, A.R. “The Sensitivity of the Yield of Personal Income Tax in the United Kingdom.” *The Economic Journal* 287 (1962), 576-596.

Schunk, D., and Porca, S., 2005. “State-Local Revenue Diversification, Stability and Growth: Time Series Evidence.” *The Review of Regional Studies* 35 (2005), 99-103.

Sen, P. “A Note on Estimating Tax Elasticities.” *ESocialSciences Working Paper Series 2328* (2009), 1-12.

Sharpe, W.F. “A Simplified Model for Portfolio Analysis.” *Management Science* 9 (1963), 277-293.

Sharpe, W.F. “Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk.” *Journal of Finance* 19 (1964), 425-442.

Smith, A. *An Inquiry into the Nature and Causes of the Wealth of Nations.* London: W. Strahan and T. Cadell, 1776.

Stein, C. “Inadmissibility of the Usual Estimator for the Mean of a Multivariate Normal Distribution. *Proceedings of the 3rd Berkeley Symposium on Probability and Statistics 1*. Berkeley: University of California Press (1955), 197-206.

Suyderhoud, J. State-Local Revenue Diversification, Balance, and Fiscal Performance. *Public Finance Review* 22 (1994), 168-194.

Tobin, J. “Liquidity Preference as a Behaviour Towards Risk.” *Review of Economic Studies* 67 (1958), 65-86.

Tsay, R.S. *Analysis of Financial Time Series.* New York: John Wiley & Sons, 2002.

White, F.C. “Trade-Off in Growth and Stability in State Taxes.”*National Tax Journal* 36 (1983), 103-114.

Williams, W.V., Anderson, R.M., Froehle, D.O., and Lamb, K.L. “The Stability, Growth and Stabilizing Influence of State Taxes.” *National Tax Journal* 26 (1973), 267-274.

Wolswijk, G. “Short- and Long-Run Tax Elasticities: The Case of the Netherlands.” *European Central Bank Working Paper Series* 763 (2007).