An Agent Based model of Banking Regulation

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1 Introduction

The central question that this research project aims to investigate is whether the incentives created by regulation and institutions that surround a financial market can cause financial fragility. The main driver in this model will be the incentives created for risk-taking by institutional features of the banking sector, and the evolutionary implication of these incentives on the aggregate functioning of the economy as a whole.

As the focus is on the functioning of the banking sector, the production side of the model will be simple and linear and there will be no labour. It is thus a consumption - investment model with an exogenously given production technology and variable institutional setup, but one where banks have a productive role to play in aggregating savings and allowing geared investment in potentially productive projects.

To model the idea that institutions and policies can induce more/less patient behaviour endogenously one requires a model where lending/saving decisions are made endogenously, not because of a pre-imposed assumed impatience.

Almost none of the individual components I use here are novel, but the combination of them in the structure below is quite new, so requires focussed discusion.

1.1 Suggested Reading Approach for the Monetary Research Group:

The model is quite complicated, although the individual components are well studied in other contexts.

I use an agent based computational approach rather than an direct equilibrium approach, since the model is too large to be feasibly approximated in any specific equilibrium. Indeed, I have assumed that the equilibrium is necessarily sequential: rather than have interest rates be general equilibrium objects that always clear the market, money supply and demand does the equilibrating with the Central bank the residual party in the market. This corresponds to the modern view of central banking: the central bank sets returns on its unique asset - fiat currency that is the unit of account and medium of transactions in all periods, but does not attempt to control the quantity. Indeed, monetary aggregates are not predictable with any degree of accuracy in this model - they are only *ex post* measurable.

The paper is nowhere near complete - I will not be able to show any computational results. The bits that i would like my readers for 6 August to focus on are the following:

- the assumed structure on the asset market, specifically section 2.2 and 2.4.2. Here I set out which productive assets and which derivative financial assets populate the model and how they are related and accessed by the different agents (this is largely where my accounting background is exploited, and I have not seen near equivalent treatments in the literature - hence I do not know if I left something crucial out.)
  - I leave the solution of the agent's problem in general "Value function" form - without any indication of how I propose to solve the problem - what i would like there is simply that you check that the approach seems logically sound in principal.
- my approach to modeling insolvency
- the assumed motivations behind bank behaviour
  - How bank preferences are determined (here the specific assumtions are not that crucial to analyse, but the general nature of the assumptions - that the bank's preferences are a function of the preferences of its owners
  - How banks view loans relative to one another
- The assumed sequence of events within a period. This is the central "computational" assumption - please read this carefully to see if you can spot any holes that are likely to be problematic in interpreting this as a reasonable model that would capture institutionally important features of the financial sector.
  - The entire text i present is basically built as one large scale algorithm for a computationally simulated economy, hence all assumptions are made to facilitate "potential" computational feasibility while simulataneously allowing for features i think are important to the recent and ongoing banking crisis which would make standard equilibrium characterization or computation extremely difficult. I don't yet have much idea of how long it would take to solve a single run of the economy, but so far, I can visualize the code i have to write for all parts (it will infact read a lot like the model in text form)
  - One of the features i like about the computational approach is a reinterpretation of a "problem" with simulation: Even in a model that can be shown to have a unique equilibruim, trying to find it by Monte

Carlo simulation of actual decisions in an economy is known to be a very inefficient algorithm: it does not always converge to "the" equilibrium, and if it does, it does so very slowly. In my view this can be reinterpreted as a result: I do hold to the belief that economic equilibria exist and are always the irresistable "region of attraction" in a system that govern the macro behaviour in any economy. I also believe, which is less universally accepted, that the real world can have multiple time varying equilibria, and that institutional features in the real world economy make it difficult to identify "the" equilibrium. Hence studying the convergence properties of the model under different economic assumptions is an economic result, not merely a test of algorithm efficiency - it may very well be that certain institutions allow longer and more severe deviations from equilibrium adjustment paths.

The following parts have detailed expositions that are simple in principal but messy in their particulars (and not very interesting - its just algebra that isn't very "clean") - for these I suggest just skimming over the analytics and seeing if you agree with or find holes in the "motivation" or "heuristic" verbal version of the analysis:

- the specific individual utility functions and how they map into "bank utility" functions - i use a variety of averaging choices that look nasty but are the simplest in the context - these will be done computationally, and hence need not be very elegant. Focus on the assumptions behind the specific functional forms and their relations.
- banks' approach to loan approvals: i consider this from a whole sale financed cash flow perspective - this necessitates the use of "index sets" quite a lot. These are just "the list of potential clients" or the "list of owners". The "index function" that ranks alternative loans looks particularly unpleasant but is just a direct valuation (according to some fucntion) of the expected cashflow. Whenever you read an object $\sum_{i \in A}$ just interpret it as looking up who is in set A (it is a list of "names") and add their quantities together and ignore the equivalent quantities of those not in A.
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The following parts are still mostly unfinished:

- How banks choose deposit and lending rates - in principal this will happen by a very simple way that is similar to the choice of loan approval rule - i.e. the information constrained maximization of some valuation of the gross return on equity of a bank - but the details of what is known and projected about the period for which the rates must hold is still to be specified.
- The monetary policy problem to be solved by the central bank (I give a simple version, but I am not happy with it yet)

- The monetarist arithmetic - in section 3.1 and 3.2 - feels “counter intuitive” but i cannot get away from the fact that it is sound from an accounting perspective. Please pay special attention here (“the money supply” is a negative quantity in the central bank’s resource constraint, which I struggle to close off with the rest of the economy.

2 Model

The model is populated by a large but finite population of N of economic decision making agents that must make consumption, saving/lending and entrepreneurial decisions (from hereon just “agents”). Agents in this model will endogenously choose to be bank equity holders, deposit holders or entrepreneurs depending on individual circumstance and regulatory environment.

A crucial component is the explicit derivation of bank objectives and knowledge: Since some agents will choose to be bankers, the goals of and information sets available to banks should be some aggregation of the preferences and information of the individuals that run it.

The population is partitioned into M regions. Each region has a local bank that sells equity, takes deposits,invests in loans to agents and trades in the interbank market. We will restrict the relationship between banks, depositors, equity holders and lenders across regions in different ways below, to model specific cases of interest. I will present here only the most restricted case I will consider in this research project: agents can only deal with their own regional bank, but regional banks can trade with each other and the central bank.

Imposing regional banks that are few in number relative to the number of agents allows for an interesting role for banks and an interbank market with systemic risk. In later versions of this model, I will allow self selection of clients to banks, which will most probably have interesting implications along adverse selection lines, but for now I will impose the partitioning exogenously to focus on other elements of analysis.

2.1 Notational Conventions

Superscripts will be used to identify different financial instruments. Subscripts will be used for bank/region, individual/asset and time indices. Banks/regions will be indexed by subscripts k or h while agents/assets will be indexed by subscripts i or j. Exponents will be indicated on the outside of parentheses where confusion might arise.

The rate of return on any instrument will be indicated with the letter R. Upper case refers to gross rates: R_t^z is the gross rate on instrument z in period t; lower case letters represent net rates ($r_t^z = R_t^z - 1$). I will mostly use gross rates in the exposition below.

Since there will be uncertainty and learning both by individuals and firms (banks), I need notation that deals with the distinction. The belief of an agent will be indicated with a single hat, while the belief of a bank will be indicated by a "double hat": e.g. $\widehat{\zeta}_i$ is the belief of agent i and $\widehat{\widehat{\zeta}}_k$is the belief of bank k about the value of unknown quantity ζ.

Since $e_{k,i,t}$ will denote agent i's holding of equity in bank k and $E_{k,t}$ will denote the total equity capitalization of bank k, the expectation operator is denoted as $\mathbb{E}$ and the exponential function as e.

2.2 Financial Instrument Space

The central modelling feature of this paper is that there is an exogenously given set of real (productive) assets, but only an endogenous subset of the available assets is invested in (i.e. the asset market is in a sense "endogenously incomplete"). All other assets are "pure financial instruments". That is: loans, deposits, bank equity or any other instrument in the asset market are derivatives of the underlying set of assets invested in. The return on derivative assets will thus be a function of which underlying assets are actually invested in.

To be precise: In my model there is a fixed set of *individual specific* productive assets available (each an exogenously given stochastic process with strictly positive returns). Moreover, these assets are the *only* production technology available. All current consumption and investment must be sourced out of the returns from past investments by agents in their individual technologies. I think of "real investment" here as all investment whether physical or human capital that has some random positive outcome independently of the financial or monetary system.

A key novel feature of my model (as far as my reading has allowed me to judge) is that only some agents will invest in their individual specific asset; others will invest only in financial instruments. The model is set up so that agents only choose to invest in their own unique technology if they believe it a productive and safe enough investment to have in their portfolio given the alternatives: the set of financial assets available from banks. Agents thus face a standard optimal portfolio problem where the risk-return structure of the equity, deposits or any other liability of a specific bank is directly dependent on the portfolio of loans and other assets that the bank chooses to hold. The performance of the loanbook depends on the performance of investments in the real productive assets that the loans finance. In sum: what is novel in this model is that economic performance in aggregate (the cyclicality and growth in the income distribution) is an endogenous consequence of the lending and financing decisions made by banks. These decisions are in their turn shaped by the incentives created by financial regulation and monetary policy.

I use "individual specific assets" as the foundation of this model based on the simple accounting identity that must hold in a closed system: the real GDP is the sum of (some real valuation of) all production in that takes place, which must be the sum over individuals of production attributable to the efforts and resources invested by each individual. So (more likely that it only really applies to human capital - but since investments mature within each period, the distinction is moot.)

I will explicitly derive the relationships between the real and derivative assets below, but for now I just list all assets to fix notation. Note that all investment levels and rates of return are in nominal terms.

Agent i can:

- invest $a_{i,t}$ in her own individual specific **asset** with stochastic gross return $R^a_{i,t}$;
- invest $e_{k,i,t}$ to buy a share of the **equity** of bank k, with stochastic gross return $R^e_{k,t}$;
- invest $d_{k,i,t}$ in a **deposit** at bank k, with gross return $R^d_{k,t}$(this will possibly be stochastic, possibly be safe - I will consider different underlying institutions that will result in one or the other).

On the liability side, agent i can, subject to her loan being approved,

- borrow $l_{k,i,t}$ from bank k, where the **loan** must be repayed at gross rate $R^l_{k,t}(g)$, conditional on the lender not being insolvent, which will occur with strictly positive probability in all computed equilibria of this model. I allow the rate to depend on the level of gearing g for realism and computational reasons. Note that this rate may be specific to a bank and is a function of the gearing of the loan, but not on the identity individual borrower.
- Given that a loan will not be repaid at its promised rate $R^l_{k,t}(g)$ with certainty, additional notation is needed to characterize bank decisions and outcomes. Since the outcomes for any specific loan is conditional on the performance of the other assets of the borrower, the actual as well as the "anticipated" gross return on any loan *is* a borrower specific random variable:
  - The actual random gross return per dollar loaned to agent i is dentoted $R^l_{k,i,t}$; however,
  - since the bank is not perfectly informed, its *anticipated* random gross return per dollar loaned to agent i is denoted $\widehat{\widehat{R^l}}_{k,i,t}$
  - The distribution of each of these is derived in section 2.6.2, and differ only in that the "true" parameter that governs the productivity of agent i is not known with certainty. I.e. the bank holds a belief about the productivity of the person it is lending to that induces an *anticipated* distribution on the likely returns from a loan. This anticipated distribution differs from the actual if the belief is inaccurate.

A bank will, of course, be the counterpart in each of the equity, deposit and loan instruments above. Additionally, banks have access to the interbank (money) market as well as the central bank as fundamental provider of fiat currency. In order for the interbank market to have an interesting function, it is necessary for the Central Bank to also maintain a interest differential: It must pay a lower return on reserves kept by commercial banks at the Central Bank than it charges on reserves lent out. When there is possibility of confusion I refer to "commercial banks" as opposed to the Central Bank.

- a bank may borrow cash **reserves** from the central bank at the repo/policy rate $R^{\overline{r}}$.
- a bank may keep cash reserves at the central bank, at gross return $R^{\underline{r}} < R^{\overline{r}}$. This is the equivalent to "notes and coins" liability of the central bank, so the nominal return will probably be normalized to zero (as e.g. in Reis, 2013).
  - Since in any period some banks will hold reserves at the bank and others will lend from the bank, I occasionally need to refer to the rate at the central bank without specifying which one is relevant to an arbitrary bank. For this purpose I will use R^r to mean:

$$R^r = \begin{cases} R^{\underline{r}} & \text{if referring to reserves kept at the central bank} \\ R^{\overline{r}} & \text{if referring to a loan from the central bank} \end{cases}$$

- bank k borrows from or lends to bank h at **money**[1] market rate $R^m_{h,k}$

Note that the last three points together imply restrictions on possible outcomes in the interbank market: The central bank is assumed never to default since it can autonomously expand its budget constraint. This means the lowest guaranteed return on any financial asset is $R^{\underline{r}}$. Its policy rate is the rate at which a bank with a short term cash deficit can borrow freely, hence no bank will pay more than $R^{\overline{r}}$ for funds on the interbank market.

In order to solve for outcomes in the interbank market, I will assume a random matching process combined with Nash bargaining governs interactions (see section 2.7), but it is immediate that at any period the following must hold: $R^{\underline{r}} < R^m_{h,k} < R^{\overline{r}}$

To allow for inflation to provide the balancing force between real and monetary aggregates, I assume that the real resources generated out of savings or investments in period $t-1$ was sold for cash at aggregate price level P_{t-1}, and can be converted into consumption or investment goods in period t at aggregate price level P_t.

2.3 Policy Space

The potential policy space in this model is quite rich, consisting of:

- Either an official policy interest rate pair $\{\ R^{\underline{r}},\quad R^{\overline{r}}\ \}$ (where the superscript r stands for repo-rate) or fiat currency money supply M^0

[1] I call the interbank market a "money" market purely to be able to use m as an identifying superscript

- in this model, these two cannot vary independently (see section 2.8.1)

- a capitalization requirement on banks, initially specified as a required equity capitalization ratio q, and
- a regulation on insolvency settlements: a fraction δ of a borrower's wealth is attachable when in default.

The goals of the central bank will be explored carefully in future work. To fix ideas for now, assume that the central bank aims to use the repo rate and/or money supply to attain a specified price P^*_{t+1} in the next period conditional on current price level P_t i.e. to attain a gross inflation rate $\Pi^*_{t+1} = \frac{P^*_{t+1}}{P_t}$

I will initially restrict some of these to be strictly exogenously fixed, but will in the end product of the thesis fully explore the most insteresting/relevant combination of these three policy instruments. Since there is no direct, standard macro policy related to δ i will probably leave it exogenous and only consider sensitivity analyses of results related to this parameter.

2.3.1 Modelling insolvency outcomes and policy

The assumption on the consequences of insolvency is a compromise between realism and modeling simplicity. It is known that limited liability induces bottom truncated utility possibility outcomes which tends to lead to corner solutions and model instability (I will write a bit more on this eventually - i do not really know how well these results are established).

To keep things smooth in this model, I make a harsh assumption on the nature of insolvency laws: i assume that when an agent is insolvent the bank can at most recover a fraction δ of the wealth of the borrower in default. This means if the the contracted repayment is L and the agent has random wealth realization W, both the bank and the borrower correctly anticipate in their optimization problems that the true repayment B will be non-linear in the wealth outcome:

$$B = \min\{\delta W, L\}$$

This has the double benefit of allowing for partial default while remaining easily modelled from both sides of the loan agreement. The repayment is piecewise defined but continuous (the first derivative of the repayment will of course be discontinous.

2.4 Agents

Agents are standard risk averse expected utility maximizing individuals with a somewhat non-standard set of assets to invest in. For now, consider this to be a standard infinitely lived agent. I will introduce life cycle and over lapping generations type effects of a limited form as a generalisation and computational simplification later.

Let $a_{i,t}$ be the resources agent i invests in her private asset, $e_{k,i,t}$ her investment in equity of bank k, $d_{k,i,t}$ the amount she keeps in deposits at bank k and $l_{k,i,t}$ the size of the loan she applies for at bank k.

2.4.1 Agent heterogeneity

For interesting results, a model such as this requires agent heterogeneity. Agents will differ from each other along three exogenous and two endogenous dimensions: Exogenously, I will start the model off with a set of agents that display a distribution of level of risk aversion and initial wealth; and live in a specific region. Decisions that agents make will determine endogenously the evolution of their wealth and the portfolio of assets they choose to hold. Eventually, I will allow beliefs to be the consequence of decisions in the past, but I do not do so yet in this version of the model.

Formally, agent i in region k attempts to maximize the present discounted value of life-time utility derived from consumption, where the utility function is a standard Constant Absolute Risk Aversion (CARA) function, subject to a budget constraint determined by the available assets in region m and the agent's initial wealth w_0^i (and beliefs, although I suppress this for now).

This implies the agent solves the following problem:

$$\max_{\{c_{i,t},a_{i,t},e_{k,i,t},d_{k,i,t},l_{k,i,t}\}} \mathbb{E}\left[\sum_{t=1}^{\infty} \beta^{t-1} u_i\left(c_{i,t}\right) | w_{i,0}\right]$$

Where

$$u_i\left(c\right) \quad = \quad -\mathsf{e}^{-\rho_i c}$$

I will not characterize this sequence based definition of the problem but rather a recursive form described in section 2.4.4.

First I describe the available assets in detail.

2.4.2 Agent Specific Assets and Financing Technologies

Each agent is endowed with an individual specific "entrepreneurial" stochastic production technology with return process $R_{i,t}^a$ that *only* she can invest in. She will do so only to the extent that she believes is optimal, conditional on the financing options available. She need not invest in this technology to save, however, as there will be a full banking system: her alternatives are to save in the form of bank deposits and/or to hold bank equity.

Every dollar agent i invests in her technology yields the following nominal return:

$$R_{i,t}^a = P_t \alpha_i \mathsf{e}^{\varepsilon_{i,t}}$$

Where α_i is an agent specific mean and $\varepsilon_{i,t}$ an agent specific shock possibly correlated over time but not across individuals, such that $E\left[\frac{R_{i,t}^a}{P_t}\right] = \alpha_i$. Initially, for computational reasons, I will keep the distribution of $\varepsilon_{i,t}$ very simple.

The other financial instruments available in period t from bank k are: equity with random return $R^e_{k,t}$; bank deposits with promissed return $R^d_{k,t}$; and bank loans with a known interest charge schedule $R^l_{k,t}(g)$ that is increasing (and possibly convex) in the gearing ($g = \frac{a}{a-l}$) of the investment project[2].

I require this complicated structure in order to study the features of the banking market related to the recent financial crisis, but this means I must make strong simplifications elsewhere in order to have a manageable model. I label these as a succession of assumptions on the structure and functioning of the asset markets in this economy.

Asset Market Assumption 1: An agent must in each period choose to invest *either* in her private asset *or* in bank equity, *but not both.* Moreover this decision must be made before the loan application is approved and cannot be reversed within the period. In the next period she can again freely choose which asset to invest in.

I need assumption 1 to keep the equilibrium in asset markets and bank's optimal decision problem manageable: the equity market will open only once in each period, and those who invest in a bank gain (some) control over the operating procedures of that bank. Thus the "goals of the bank" will be a function of the goals of the *owners* of the bank.

While this "one-or-the-other asset" is likely to be a restrictive assumption, it greatly simplifies the problem that the agents and banks have to solve (as well as the model solution as a whole), as mid-period changes in equity will make the "goals of the bank" too difficult to handle simply.

I label agents who invest in their private asset "entrepreneurs" and those that invest in bank equity "bank owners" or "equity holders".

Asset Market Assumption 2: Loans are exclusively used to finance investment in private asset.

Assumption 2 (with Assumption 1) implies that no agent borrows to finance equity. While this may seem a restrictive assumption to impose, I conjecture that it will necessarily hold in any equilibrium in this model, even if it is not imposed, for the following simple reason: bank equity returns are financed out of interest charged on loans minus interest paid on deposits and other sources of capital, i.e. equity at any bank must, on average, have a lower return than the loan rate at that bank, so it will not be optimal to borrow to buy equity. Similarly, since there is no liquidity need for deposits, no borrower will choose to

[2] This helps avoid a specific type of corner solution: with a fixed interest charge and a linear returns technology, if borrowing any amount is utility increasing, then borrowing the maximum allowed is optimal. This would mean that *all* agents that wish to borrow will maximize borrowing. This would then require an exogenous (or endogenous) borrowing constraint to be imposed for an equilibrium to exist. Assuming an increasing convex cost of borrowing ensures that desired borrowing has an interior optimum for all parameter values. Restricting attention to interior optima greatly reduces computational burden. Additionally, a gearing sensitive loan rate is somewhat more realistic than assuming a fixed loan rate irrespective of the size of the loan relative to the collateral.

hold both a deposit *and* a loan. Imposing this assumption outright reduces the complexity of the solution algorithm (without loss of generality if my conjecture is true). Deposits are purely savings instruments in this model.

I do not wish to restrict the options further: deposit holders (who do not borrow) may invest out of their own wealth in their private asset. Since deposit rates at any bank will be strictly below lending rates (at any bank) in this model, there is a disconuity between the lending and savings decisions of entrepreneurs that must be treated with discrete optimization methods.

An important operational feature of banks in this proposed model is that they choose whether or not to approve a loan based on incomplete information. That means some loans will be rejected that ought to have been accepted and some accepted that ought to have been rejected. For now, assume that agent i who wishes to make a loan believes it will be approved with probability[3] $0 < \pi^i < 1$ but that applying for a loan is costless, so that any entrepreneur that wants to borrow submits a loan request irrespective of how small she believes the probability of success is[4].

In order to reduce the complexity of the loan approval process, I make another strong assumption:

Asset Market Assumption 3: Loan applications are only approved or denied in full. No partial loans are made.

With these simplifications at hand, I can characterize the structure of the individual optimization problem that must solved by any agent.

2.4.3 Agent Beliefs

Since the only productive assets in this economy are the individual specific production technologies operated by/invested in by individuals, the only relevant beliefs to keep track of in this economy are those of the individual decision-makers on the vector of individual specific asset returns. In fully general form this would of course be impossible, but the asset structure in this economy is rather simple. We will assume that the nature of processes (practically: their functional forms) are known but parameters are not, which reduces the problem greatly, and allows it to be compactly represented in convenient matrix forms that also aid computation.

We will maintain the following assumptions in this development:

1. All agents know the population structure and can observe perfectly which assets are invested in, but imperfectly (to varying degrees) how much is invested in each or what the returns are. Heuristically: Everyone can see which people built shops, but not everyone knows how much was invested in each, nor how profitable the shop is.

[3] This probability-belief will eventually be endogenously different from the actual approval probability, but for now I keep it exogenously fixed

[4] I will explored whether it this is strategically consistent with the incentives implied by this synthetic economy.

2. All agents know the functional form of all available real assets but not their parameters

The feature that makes this model computationally feasible is that the set of available real assets is finite, fixed and exogenous. This means, in the current setup, the only objects over which there is *fundamental* uncertainty are the means of the individual specific returns processes[5]:

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_N \end{bmatrix}$$

Let the belief of agent j about the mean of the process of agent i at time t be $\widehat{\alpha}_{i,j,t}$. Then her beliefs at time t about the potential productivity of the whole economy at time t (i.e. the distribution of the mean returns of the assets across individuals) is given by:

$$\widehat{\boldsymbol{\alpha}}_{j,t} = \begin{bmatrix} \widehat{\alpha}_{1,j,t} \\ \vdots \\ \widehat{\alpha}_{i,j,t} \\ \vdots \\ \widehat{\alpha}_{N,j,t} \end{bmatrix}$$

So we can collect the relevant beliefs of the entire population about the productivity potential of the whole economy at a specific point in time in the $N \times N$ matrix:

$$\widehat{\boldsymbol{\alpha}}_t = \begin{bmatrix} \widehat{\boldsymbol{\alpha}}_{1,t} & \dots & \widehat{\boldsymbol{\alpha}}_{j,t} & \dots & \widehat{\boldsymbol{\alpha}}_{N,t} \end{bmatrix}$$

All the beliefs about process i is then the i^{th}row of this matrix. Note that it is immediate that all model consistent probabilistic views outcomes be determined by these beliefs.

Since the gathering of information about the cross-section of the income distribution is not a trivial task, even for economists who specialize in doing so, we will restrict information capacity of the individual here (similar to the solution method for macro models with heterogenous agents proposed by Krusell and Smith, 1998), mainly to gain tractibility, but also for some realism.

For the very first run, I will not allow agents to learn: I will simply assume that each agent observes the true state of the economy with some noise:

[5] Of course, the agent also does not know what any specific future realization of the error process will be, but we assume that the process that governs ε is common knowledge for now, so that beliefs are always correct and we do not need to consider how they are formed. This assumption will be relaxed in future work.

$$\widehat{\boldsymbol{\alpha}}_{j,t} = \begin{bmatrix} \alpha_1 \mu_{1,j,t} \\ \vdots \\ \alpha_i \mu_{i,j,t} \\ \vdots \\ \alpha_N \mu_{N,j,t} \end{bmatrix}$$

2.4.4 Agent Optimization Problem

An agent[6] is constrained by his wealth w_t and makes decisions about uncertain outcomes based on his information/beliefs $\widehat{\boldsymbol{\alpha}}_t$ at the begining of the period (later versions will include credit history as a contraint).

Their available information determines their beliefs about the returns of the various assets available in the period which in turn determines the optimal decision in each period.

Beginning of period nominal wealth is the consequence of the investment decisions and stochastic realizations in the previous period:

$$P_{t-1} w_t = \begin{cases} R^e_{t-1} e_{t-1} + R^d_{t-1} d_{t-1} & \text{if } a_{t-1} = 0 \\ R^a_{t-1} a_{t-1} - \min\left\{\delta R^a_{t-1} a_{t-1}, R^l_{t-1}\left(g_{t-1}\right) l_{t-1}\right\} & \text{if } a_{t-1}, l_{t-1} > 0 \\ R^a_{t-1} a_{t-1} + R^d_{t-1} d_{t-1} & \text{if } a_{t-1}, d_{t-1} > 0 \end{cases}$$

Conditional on his wealth and beliefs, the agent must make the following decisions/plans in each period:

In the begining of the period, she must choose whether to invest in the private asset or in the bank equity market. She makes this choice by constructing the following contingent plans and selecting the plan that yields the highest expected welfare:

- Plan E: Invest in the equity market with a contingent plan of optimal consumption, deposit and equity holdings.
- Plan A: Invest in the private asset
  - Plan A1: Optimal consumption, lending and investment assuming that the loan request is approved
  - Plan A2: Optimal consumption, saving and investment assuming the loan is not approved
    - Note: if Plan A2 gives higher value than Plan A1, then the agent will not apply for a loan at all. Since all three plans have to be solved for before we can make this judgement, this possibility does not complicate the characterization of the problem further

[6] I suppress the individual and regional indices and the dependence of the value functions on beliefs to keep the notation as simple as possible.

Each of these three options individually is a standard dynamic optimization problem, but their joint nature is more complicated. I conjecture that the problem as a whole will also fall into a neatly recursive form which will be exploited with some timing assumptions to solve the evolution of the model under different institutional and regulatory assumptions. This conjecture will have to be confirmed in the course of the research, of course.

I characterize these problems and the nature of their solutions in several steps using a Value Function approach cast in form of Bellman's equation.

Let the maximum value of lifetime utility obtainable with wealth w_t from Plan s be $V_s(w_t)$.

The stochastic nature of investing in equity is different from investing in assets in terms of uncertainty. I found it necessary to develop the following framework before I am able to discuss the agent's views of bank equity (in a later subsection).

Assume that the agent believes her loan will be approved with probability π.

$V_{A1}(w_t)$ is the value of the optimal private asset investment plan if the loan application is approved and is given by:

$$\begin{aligned} V_{A1}(w_t) &= \max_{a_t,l_t}\left\{u(c_t)+\beta\mathbb{E}_t\left[\max_{s\in\{A,E\}}V_s(w_{t+1})\right]\right\} \\ \text{subject to:} & \\ P_{t-1}w_t &= P_t(c_t+a_t-l_t) \end{aligned}$$

Note that any choice of (a,l)implies a gearing ratio $g=\frac{l}{a-l}$ (so $a=\left(\frac{1+g}{g}\right)l$) which in turn determines the interest rate on the loan (and hence affects expected future wealth non-linearly).

$V_{A2}(w_t)$is the value of the optimal private investment plan if the loan application is denied (or if no loan application is made) and is given by:

$$\begin{aligned} V_{A2}(w_t) &= \max_{a_t,d_t}\left\{u(c_t)+\beta\mathbb{E}_t\left[\max_{s\in\{A,E\}}V_s(w_{t+1})\right]\right\} \\ \text{subject to:} & \\ P_{t-1}w_t &= P_t(c_t+a_t+d_t) \end{aligned}$$

Given these, the expected lifetime utility value of investing in the private asset is then given by:

$$V_A(w_t)=\begin{cases}\pi V_{A1}(w_t)+(1-\pi)V_{A2}(w_t) & \text{if } V_{A1}(w_t)>V_{A2}(w_t)\\ V_{A2}(w_t) & \text{otherwise}\end{cases}$$

The value of investing in equity, $V_E(w_t)$, is given by:

$$\begin{aligned} V_E(w_t) &= \max_{d_t,e_t}\left\{u(c_t)+\beta\mathbb{E}_t\left[\max_{s\in\{A,E\}}V_s(w_{t+1})\right]\right\} \\ \text{subject to:} & \\ P_{t-1}w_t &= P_t(c_t+d_t+e_t) \end{aligned}$$

I assume that an agent chooses to invest in equity whenever $V_E(w_t) \geq V_A(w_t)$ (i.e. if there is a tie, the agent chooses equity).

I will only be able to characterize the appropriate beliefs that any agent will hold over returns to bank

The solution of this problem is not computationally trivial, and a time consuming part of the research will be constructing algorithms that can give adequate approximate solutions in reasonable time. Some strong assumptions may be necessary to make this truly feasible[7].

2.5 Timing of Events and Outcomes within a period

Conditional on a solution to each of these subproblems being found for each individual, I will use an assumed "market sequence" of events to define model consistent beliefs for the agents making the investment plan at the begining of the period and to find the aggregate consequences of these decisions. This allows the evolution of the state variables of the problem to (potentially) approach any equilibrium that exists for a given set of parameters. I do this rather than imposing an equilibrium of a certain type and and trying to characterize it for two reasons: first, in learning models it is most likely that there will be many equilibria; second, the point I wish to explore is whether there are certain types of institutions/conditions that lead to certain types of equilibria with greater probability than others.

The assumed sequence of events is as follows:

State variables that enter period t as consequences of decisions and stochastic realizations in period $t-1$ are:

- individual specific wealth and beliefs for all individuals, $\{w_{i,t}, \widehat{\boldsymbol{\alpha}}_{i,t}\}_{i=1}^{N}$
- offered deposit rates ($R_{k,t}^{d}$) and loan rate schedules ($R_{k,t}^{l}(g)$ where g is the level of gearing) for all K banks. Banks are legally obliged to hold to these within the period, conditional on not going insolvent (see step 8 and the sections that follow).

1. All agents solve their individual problems (which vary between any two agents due to differences wealth, beliefs and risk preferences) and truthfully submit their optimal plans to an impartial, perfectly mechanical, error free central clearing house. A bank at this stage is defined by a set of deposit and lending rates fixed in the previous period and a "potential pool of owners, savers and lenders".
2. The equity market opens and the clearing house completes the transactions of all those agents who chose not to submit a loan application.

[7] For instance, one very strong assumption that one might consider is that agents are excessively myopic: if they consider one of the two options, they believe they will stay in that option forever. In this case, each problem is trivial to solve computationally, as it is entirely standard. Alternatively I am considering a finite lifetime economy, which gives a clearly defined terminal asset value (0) that can be used to obtain solutions by backward induction.

Banks now have owners, know their equity capitalization and have a first round idea of their deposit liabilities. Banks will condition their loan approval rules on this information only. (At this stage, the clearing house does not send the loan requests to the banks).

3. Bank owners/managers determine and commit to a rule for loan approval that internalizes any regulatory requirements and institutional features (e.g. capital adequacy ratios, central bank reserve requirements, implicit bail-out guarantees etc.), before receiving information on the loan requests (details will be given in the banking subsection). This assumption is strong, but is intended to heuristically capture the idea that operating procedures in banks are likely to be institutionally fixed over periods in which many loan applications are received. In the real loan market, while loan applications occur spread out over a year, the operating procedures used in the bank to evaluate each loan are not revised after every loan decision.

4. Banks receive all loan applications and concommittant information about the underlying individual specific assets. Loans are approved or denied strictly according to the rule chosen in step 3. Until this point, nothing ensures that banks' inflows and outflows will balance. Some banks may have surplus available funds, others shortages conditional on the individual supplies and demands of their owners and customers.

5. The interbank market opens and clears. The nature and functioning of the interbank market is discussed in section 2.7.

6. Approved loans are paid out to borrowers and all agents consume and invest in the various assets as planned, and shocks realize. Let $\mathcal{A}_t := \{i \,|a_{i,t} > 0\}$ be the index set of assets actually invested in.

7. Central Bank observes outcomes, sets policies $\left[R^{\underline{r}}_{t+1}, R^{\overline{r}}_{t+1}\right]$for the following year.

8. Bank owners set deposit and loan rates for the following year, all assets are liquidated and yields paid out according to available funds and insolvency laws.

9. Agents wealth and beliefs are updated.

What remains to be specified before I turn to computational matters, is the operation of banks (for the various decisions they have to make) and the updating of beliefs of agents.

2.6 Banks

In my opinion, the minimum components that must be allowed for in any model that wishes to study the institutional, regulatory and monetary policy related causes of the recent banking crises are the following:

- banks must have a productive function in the economy so that decisions made by those who run banks affect the aggregate real outcomes in the economy; and
- the instutional, regulatory and monatary policy features of the economy must *endogenously* affect the incentives of agents that manage banks. For this purpose, banks in the model must make autonomous, non-equilibrium governed, discreet approval decisions on loans that may default. Put differently: banks must be able to make "mistakes" in which loans are approved and which disapproved. This is the only way that there can be an endogenously positive probability that the bank will default unless there are back stop measures like deposit insurance.
- A functional interbank market that allows
  - wholesale funding on the interbank market as a main source of working cash flow.
  - systemic risk to develop due to endogenous interdependencies between banks
  - a modern run on a bank when it cannot borrow on the interbank market. (this part cannot not yet occur - the mechanism that allows this to happen is still to be thought up)

To achieve the first, I make the stark assumption that it is prohibitively expensive for an agent with relatively low own productivity but relatively high available resources to find and trust a more productive agent enough to invest with him. Banks provides "counterparty insurance" and economies of scale by aggregating and centralizing the process. In other works banks serve as cost effective deposit aggregators and providers of gearing to agents in control of productive technologies. I am thus implicitly assuming that banks can (costlessly) enforce repayments of loans subject to the insolvency laws given above.

Recall that banks are indexed by k or $h \in \{1, ..., K\}$ (I use the convention when describing the decision problem facing a typical bank that the "decision making bank" is k and "possible counterparty" banks are indexed by h).

This makes it convenient to label the Central Bank as bank "0" so that $m_{0,k}$ is the debt/asset position of bank k at the central bank

I make some timing assumptions that serve to turn the bank problem into a static one whose funding is wholesale: i.e. loanbook is funded (at the margin) out of the interbank market and the reserve market run by the Central Bank. The static nature derives from the fact that I only allow single period maturity instruments. In future versions, I will build in a functional maturity structure.

2.6.1 Bank Preferences and Information Sets

A near universal assumption in economic theory is that large firms in uncertain environments act as expected profit maximizers. This assumption is equivalent to assuming risk neutrality, as the implied objective function is a linear function of possible profit realizations. A common motivation (beyond that of analytical ease) is that large firms can more readily access fair insurance and hence will in equibrium be fully insured and act like a risk neutral firm.

In this model, this assumption is undesirable for two reasons: first, all banks will be subject to aggregate risk that cannot be insured or diversified away; second, banks are *owned and controlled* by risk averse expected utility maximizers.

Assuming that banks are "passive" expected profit maximizers does not allow one to build in behaviour such as "over optimism" in any satisfactory way: profit maximizers have linear utilities that are entirely determined by the expected value of any random gamble. At the cost of significantly more complicated modeling, I choose a different route, largely motivated by my personal belief that no banker will keep his job if he claims or acts as if he did not care about variance (at least). What we need is a theory of where the risk preferences of banks originate, ideally one where the risk apetite endogenously evolves from the conditions and regulations in the market.

I start off with a very simple and still unrealistic assumption that is at least a step away from simple profit maximization:

I assume that the preferences that drive a bank's decisions are some aggregation of the preferences of the owners of that bank, using some function of their proportional shareholdings. For the purposes of my longer term research agenda, I will begin by investigating several exogenously imposed aggregation rules; I will not consider endogenous, stragetic or collusive behaviour among subsets of shareholders, nor will I consider principal-agent problems in the management of banks. These are undoubtedly important in analizing incentive effects and consequent financial market fragility, but are too complicated in an already busy model.

For the purposes of this proposal I stick to the most "optimisticly neoclassical" of assumptions about the internal functioning of banks:

Bank Assumption 1: The value function $v(.)$ that represents the preferences of a bank is an equity-share weighted (geometric) average of the utility functions that represent the preferences of its equity holders[8], and the objective of the bank is to maximize the valuation of gross return on equity.

Banks form and disolve within a period, so this is a static problem constrained by policy and market conditions.

Since I will use standard CARA utility functions in the first round of the model, a geometric average is computationally more tractible than a linear aver-

[8]Even excluding strategic, collusive and agency problems, there are a number of reasonable alternatives that one can use, such as a majoritarian assumption that only the preferences of the largest shareholders who collectively own greater than 50% of equity are aggregated in the preferences of the bank. These simple alternatives will be easy to add as counter factuals after the computational algorithm of the model is complete.

age. I will eventually be able to test the sensitivity of results to these exogenously imposed assumptions.

Formal construction[9] of the objective function of bank k Let the index set of agents that hold strictly positive levels of equity in bank k in period t be $\mathcal{E}_{k,t} := \{i \,|e_{k,i,t} > 0\}$. Then the share of equity that is held by agent i is:

$$s_{k,i,t} = \frac{e_{k,i,t}}{\sum_{j\in\mathcal{E}_{k,t}} e_{k,j,t}}$$

and the objective function[10] that represents the preferences of the bank over uncertain outcome z is:

$$v_{k,t}(z) = -\prod_{i\in\mathcal{E}_{k,t}} |u_i(z)|^{s_{k,i,t}}$$

Note that this function will necessarily be concave (i.e. imply risk aversion) if all the constituent functions are concave (which they are by the assumptions above, although there is actually nothing stopping me from populating the model with some risk loving individuals).

While this seems messy, it boils down to a very simple form in practice. For instance: if agents 1 and 2 are the only equity holders of bank k and they hold equal shares, then the objective function of the bank is

$$\begin{aligned} v_k(z) &= -\left(\mathsf{e}^{-\rho_1 z}\right)^{\frac{1}{2}}\left(\mathsf{e}^{-\rho_2 z}\right)^{\frac{1}{2}} \\ &= -\mathsf{e}^{-\left(\frac{\rho_1+\rho_2}{2}\right)z} \end{aligned}$$

I.e. the effective coefficient of absolute risk aversion of the bank, $\rho_{k,t}$, is the share-weighted linear average of the risk aversion coefficients of the share holders:

$$\rho_{k,t} = \sum_{i\in\mathcal{E}_{k,t}} s_{k,i,t}\rho_i$$

When I characterize the bank's problem, I will suppress most of the super and subscripts I use here, but these definitions hold throughout what follows.

The information available to the bank (or equivalently, the beliefs of the bank) can also have several equally convincing forms. I have not explored this feature in detail and will in the course of the research evaluate the relevant strands of the literature in economics (specifically information economics and industrial organization) and decision theory in order to be able to choose how to

[9]Note that this construction is only valid for exponential CARA individual utility functions whose sign conventions are a little different from the standard CRRA utility function. An equivalent construction for any standard utility function will not be difficult to derive.

[10]The absolute value operators employed in this definition are without loss of generality. They are necessary in the definition of $v_t^k(z)$ to ensure that the value function is strictly increasing.

most appropriately model this feature. For now, I make a parallel assumption to assumption on the preferences of the bank.

Bank Assumption 2: The beliefs of a bank are an equity-share weighted (linear) average of the beliefs of its equity holders.

Recall that the beliefs of agent i about the distribution of productive potential of the economy is collected in the N-vector $\widehat{\boldsymbol{\alpha}}_t^i$. When considering the decision problem of the bank I therefore assume that the belief of bank k is (with double hats to distinguish it from the belief of an individual agent):

$$\widehat{\widehat{\boldsymbol{\alpha}}}_{k,t} = \sum_{i \in \mathcal{E}_{k,t}} s_{k,i,t} \widehat{\boldsymbol{\alpha}}_{i,t}$$

2.6.2 Deriving the optimal loan approval rule

Denote the set of applications for loans from bank k in period t by $\widetilde{\mathcal{L}}_{k,t}$.

Recall from the timing and asset market assumptions that the loan approval process for period t is committed to when the bank has the following information:

- It knows the distribution of the random component of gross productive asset return (all agents are assumed to know this with certainty)
- The official capitalization requirement is q_t (expressed as ratio of equity to liabilities[11] and set by the Central Bank in the previous period)
- Its equity capitalization is $E_{k,t} = \sum_{j \in \mathcal{E}_{k,t}} e_{k,j,t}$
- the promised deposit rate $R_{k,t}^d$ and loan rate schedule $R_{k,t}^l(g)$

When it actually recieves the loan applications, the bank will also know

- the identities of all loan applicants and its beliefs about their productivity levels (which are the corresponding of elements of $\widehat{\widehat{\boldsymbol{\alpha}}}_{k,t}$).
- the set of individually desired gearing ratios $\{g_{j,t}\}_{j \in \widetilde{\mathcal{L}}_t^k}$ (which determine the loan rate) and requested loan sizes $\{l_{k,j,t}\}_{j \in \widetilde{\mathcal{L}}_{k,t}}$(and hence the sizes of the proposed investment and distributions of gross returns)
- the amount of deposits $\{d_{k,j,t}\}_{j \in \widetilde{\mathcal{L}}_{k,t}}$ that will be recieved for each loan that is rejected

[11] If the requirement is in terms of cash reserves/non loan assets (which must of course be denoted M_t^0 or something similar!!!) relative to loan book, the problem would be quite different, but still manageable. I will turn to this in the near future if it seems more reasonable. for now we assume that the bank holds no cash reserves at all

Given the fixed loan rate schedule, $R^l_{k,t}(g)$ and deposit rate $R^d_{k,t}$ which are uniformly applied across all loans, the loan application from agent j, from the point of view of a bank, is entirely characterized by the vector $\left[\begin{array}{cccc} g_{j,t}, & l_{k,j,t}, & \widehat{\widehat{\alpha}}_{k,j,t}, & d_{k,j,t} \end{array}\right]$ - i.e. the gearing and size of the requested loan (which together give the size of the investment $a = \left(\frac{1+g}{g}\right) l$), the beliefs of the bank about the mean productivity of the applicant, and the deposits that would be received from the agent should the loan be denied.

The legally permitted maximum loanbook size $\overline{L}_{k,t}$ (for bank k in period t) is completely determined by its equity capitalization and the capitalization requirement set by the central bank:

$$\overline{L}_{k,t} = \frac{E_{k,t}}{q_t}$$

I argue below that it will be optimal for the bank to view the available loanbook as a resource pool that should be sequentially awarded until exhausted, starting from the "best" application and working down until the "worst" application is found whose approval will provide an increase in expected value of gross return on equity.

What I need to derive therefore, is ordering induced on $\widetilde{\mathcal{L}}_{k,t}$ by the assumed preferences and information for an arbitrary bank in an arbitrary period.

The total cashflow of a bank Before describing the evaluation of an individual loan by the bank, it is worthwhile to consider what the outcome of *any* arbitrary rule will be. This will make the derivation of the bank operational procedures reasonably straightforward. I drop time subscripts and bank superscripts as the balance sheet is considered as a static outcome for an arbitrary bank. Since the bank has no physical assets and forms and unwinds every period, the cash flow and balance sheet are virtually identical.

The outcome of any decision rule on loan approval will result in a subset $\mathcal{L}$ of the set of applications $\widetilde{\mathcal{L}}$ being approved and the rest denied. Call $\mathcal{L} := \{i \,|l_i > 0\}$ the loanbook and $\mathcal{D} := \{j \,|j \notin \mathcal{L}\}$ = the depositbook (I am allowing deposits to be zero, it is just convenient to use only one main set and its complement).

The total inflow of funds from equity is given by $E = \sum e_i$, the total inflow from deposits: $D = \sum d_i$ [fix these to show complementary nature if necessar].

The total outflow of funds for loans is given by

$$L = \sum_{i \in \mathcal{L}} l_i$$

.

Nothing so far ensures that inflows match outflows. If they do not, the residual is borrowed or offered on the interbank market or from the central bank as described in sections 2.7 and . I refer to the union of the interbank market and the facilities provided by the central bank simply as "the money market".

I still have to finalize notation for interbank transactions. This bit might need to be recast slightly

Here is where the timing assumptions made above have bite (and hence are restrictive, although not much): Banks form in the begining of a period and unwind at the end of the period. Therefore, in each period, we have to consider cashflows at the begining and end of the period. I have assumed that the loan approval rule must be chosen before the loan applications arrive and before the interbank market opens. This means that a bank cannot perfectly predict what size its loan book will be, i.e. whether it will have a surplus or deficit of cash.

Formally, I define the **initial money market cashflow** of a bank to be $\widetilde{M} = E + D - L$. Note that while the final size of the loanbook (and hence initial cashflow) is unknown at the time the loan approval rule is chosen,

1. the domain of the initial money market cashflow is known, since the domain of size of the loanbook is:$L \in \left[0, \overline{L}\right] \Leftrightarrow \widetilde{M} \in \left[E + D - \overline{L}, E + D\right]$; and,

2. a specific loan's acceptance will have a perfectly predictable (i.e. *deterministic*) impact on the cashflow of the bank. I show this carefully in the next section - remember that the loan approval rule is chosen to be conditioned on the observed characteristics of a loan, so that the cashflow implications of any loan are known when the rule is applied.

Furthermore and crucially: since I assume a "random matching and Nash bargaining" mechanism to clear the money market (see section 2.7) and that the money market opens and clears *after* the loan approval rule has been applied, a bank cannot predict how it will fund a deficit (in terms of split between interbank market borrowing and central bank borrowing) or invest a surplus (between interbank investments and keeping cash at the central bank) when it is considering approving a loan. This means that the final per dollar gross return to investing in (or the cost of funding from) the money market is a *random variable,* denoted $\widetilde{R}^m$, when the loan approval rule is executed. Since a bank can always deposit at the Central Bank at rate $R^{\underline{r}}$ or borrow from it at rate $R^{\overline{r}}$, it knows with certainty that $\widetilde{R}^m \in \left[R^{\underline{r}}, R^{\overline{r}}\right]$.

Putting the above points together means that the **final money market cashflow** that occurs when the bank unwinds is $\widetilde{R}^m \widetilde{M}$.

Now I am in a position to define the underlying composition of the **anticipated gross return on bank equity**; this is the fundamental variable of interest from the point of view of the owners of the bank: when a bank unwinds the net final money market cashflow (which may be an in or outflow) is added to the inflow of cash from loans repaid and the outflows of cash to pay out deposits with their promised returns. The residual that remains is the value of the bank that is payed out to the equity holders.

I denote the **anticipated gross return on bank equity** as $\widehat{\widetilde{R}^e}\left(\mathcal{L}\right)$, where I have made only the dependence on the set of approved loans explicit, to focus attention on how the loan approval rule determines $\mathcal{L}$ and hence equity returns in a standard optimization sense.

The individual approved loan to agent i in $\mathcal{L}$ each has a unique returns process from the point of view of the bank, which I describe in detail in the following section. For now, let the bank anticipate that loan i will have random return[12] $\widehat{\widehat{R^l}}_i$. All deposits carry rate R^d and the final money market cashflows is $\widetilde{R}^m \widetilde{M}$, so the bank's own anticipation of its gross return on equity is given by

$$\widehat{\widehat{R^e}}(\mathcal{L}) = \frac{\sum_{i \in \mathcal{L}} \widehat{\widehat{R^l}}_i l_i - R^d D + \widetilde{R}^m \widetilde{M}}{E}$$

This is random due to the loan processes and the money market terms - loans to a bank are exposed to that bank's liquidity/equity remaining positive (this sentence is old, should be adjusted to the new thingy).

Note that this allows for a bank to go insolvent in a "naturally defined" way: A bank is technically insolvent if its actual equity value $R^e(\mathcal{L}) E$ turns out to be negative:

$$R^e(\mathcal{L}) E < 0 \iff R^d D - \widetilde{R}^m \widetilde{M} > \sum_i R_i^l l_i$$

The aggregate consequences of bank insolvency will depend on the institutional setting and policy reaction. For instance, regulation may state that that depositors are highest tier liabilities and equity holders the lowest. Then the moneymarket lenders would have to accept a "haircut" on their loans to the defaulting bank.

Since banks form and disband within a period, this "insolvency" can have no disruptive, resource destroying implication (As would be the case if a bank was liquidated and could never form again. This is eventually where the model will be made more realistic to allow crises to develop.

This version, the only impact of insolvency is that the owners of the bank lose some or all of their equity. this is almost identical to a entrepreneurial agent that experienced an extremely adverse productivity shock.

The cashflow consequences of approving an aribitrary loan: Recall that, conditional on lending and deposit rates, an arbitrary loan is characterized by four numbers: $\left[\begin{array}{cccc} g, & l, & \widehat{\widehat{\alpha}}, & d \end{array}\right]$.

I assume that the bank uses the following "rule" to think about its potential loanbook: a loan application is considered to be rejected until proven worthy of acceptance.

This means that the status quo that the bank considers to be "the base scenario" is that a loan application $\left[\begin{array}{cccc} g, & l, & \widehat{\widehat{\alpha}}, & d \end{array}\right]$ implies an initial inflow of d currency units (since a loan application also contains the deposit that the applicant will make if the application is denied and the bank starts off under the presumption that the loan is denied).

If this loan is approved, the act of approving it also reverses the initial inflow of the deposit, so causes a total initial cash *outflow* (relative to the status quo) of $(l + d)$.

[12]The bank believes that the random return is governed by a fully specified anticipated distribution that internalizes all aspects of insolvency law that I derive below

Turning attention to final cashflow:

If the loan had not been approved, the deposit recieved instead would have implied a final cash outflow of $R^d d$. So:

1. approving the loan reduces anticipated final cash outflows by $R^d d$. (or equivalently, increases anticipated final cash inflows by this amount).

Once the investment is made by the borrower and the stocastic return realized, the repayment of the approved loan will

2. generate an anticipated final cash inflow of $\widehat{\widehat{R^l}} l$.

However, the initial outflow of $(l + d)$ will

either:

- decrease the cash surplus that would have been invested on the money market, i.e. *decrease* anticipated final cash *inflows* by $\widetilde{R}^m (l + d)$; or
- increase the cash deficit that must be financed on the money market, i.e. *increase* anticipated final cash *outflows* by $\widetilde{R}^m (l + d)$.

Both these effects boil down to the same point:

3. approving the loan implies an effective increase in anticipated final cash outflows of $\widetilde{R}^m (l + d)$.

Thus, putting the three points above together, the implication of approving arbitrary loan characterized by $\begin{bmatrix} g, & l, & \widehat{\widehat{\alpha}}, & d \end{bmatrix}$ is a net change in anticipated final cash flow of :$\widehat{\widehat{R^l}} l + R^d d - \widetilde{R}^m (l + d)$.

It is the *valuation* of this net change in cash flows that determines the whether a bank prefers one loan application over another.

Deriving the preference ordering on the set of loan applications received:

Recall assumptions above are made so that the objective of the bank is to maximize the valuation of gross equity returns (or equivalently to maximize the "subjective bank utility" of the market value of the bank).

The optimal loan approval rule is one that selects the most prefered subset of loans$\mathcal{L}$ from the set of loan applications $\widetilde{\mathcal{L}}$. Let $\mathcal{L}^{'}$be an arbitrary subset of $\widetilde{\mathcal{L}}$, then the most preferred subset $\mathcal{L}$ is such that[13]:

[13] Note that at the moment the loan approval rule is chosen, the level of equity is fixed, so that

$$\arg\max \mathbb{E}\left[v\left(\widehat{\widehat{R^e}}(\mathcal{L})\, E\right)\right] = \arg\max \mathbb{E}\left[v\left(\widehat{\widehat{R^e}}(\mathcal{L})\right)\right]$$

$$\begin{aligned}\mathcal{L} &= \arg\max_{\mathcal{L}'\subseteq\widetilde{\mathcal{L}}} \mathbb{E}\left[v\left(\widehat{\widehat{R^e}}\left(\mathcal{L}'\right)E\right)\right]\\ s.t. & \\ \sum_{i\in\mathcal{L}'} l_i &\leq \overline{L}\end{aligned}$$

We are assuming CARA utility ($v(z) = -e^{-\rho z}$) initially, but any function that is multiplicatively separable in the same way would work.

Note the following rearrangement (I add and subtract the cumulative final cashflow impact of all approved loans on deposits and moneymarket positions):

$$\begin{aligned}\widehat{\widehat{R^e}}(\mathcal{L})E &= \sum_{i\in\mathcal{L}} \widehat{\widehat{R^l}}_i l_i + \widetilde{R}^m\widetilde{M} - R^d D\\ &= \sum_{i\in\mathcal{L}}\left(\widehat{\widehat{R^l}}_i l_i + R^d d_i - \widetilde{R}^m(l_i+d_i)\right) + \widetilde{R}^m\widetilde{M} - \sum_{i\in\mathcal{L}}\left(\widetilde{R}^m(l_i+d_i)\right) - R^d D - \sum_{i\in\mathcal{L}}\left(R^d d_i\right)\end{aligned}$$

If we collect the "aggregate situation if no loan is approved" as

$$Z = \widetilde{R}^m\widetilde{M} - \sum_{i\in\mathcal{L}}\left(\widetilde{R}^m(l_i+d_i)\right) - R^d D - \sum_{i\in\mathcal{L}}\left(R^d d_i\right),$$

we can state

$$\begin{aligned}v\left(\widehat{\widehat{R^e}}(\mathcal{L})E\right) &= -\mathsf{e}^{-\rho\left(\sum_{i\in\mathcal{L}}\left(\widehat{\widehat{R^l}}_i l_i + R^d d_i - \widetilde{R}^m(l_i+d_i)\right)+Z\right)}\\ &= -\left[\prod_{i\in\mathcal{L}}\mathsf{e}^{-\rho\left(\widehat{\widehat{R^l}}_i l_i + R^d d_i - \widetilde{R}^m(l_i+d_i)\right)}\right]\left[\mathsf{e}^{-\rho(Z)}\right]\\ &= -\prod_{i\in\mathcal{L}}\left|v\left(\widehat{\widehat{R^l}}_i l_i + R^d d_i - \widetilde{R}^m(l_i+d_i)\right)\right|\cdot|v(Z)|\end{aligned}$$

This means we can write the anticipated *ex post* marginal value impact of approving loan application i as:

$$v(\mathtt{l}_i\,|\mathtt{l}_i \text{ approved}) = -e^{-\rho\left(\widehat{\widehat{R^l}}_i l_i + R^d d_i - \widetilde{R}^m(l_i+d_i)\right)}$$

Where $\mathtt{l}_i := \begin{bmatrix} g_i, & l_i, & \widehat{\widehat{\alpha}}_i, & d_i\end{bmatrix}$. Note that this valuation is a random variable. In order to obtain an *ex ante* ranking over the set of loan applications, I consider the expected valuation of the value of the bank: $\mathbb{E}\left[v\left(\widehat{\widehat{R^e}}(\mathcal{L})E.\right)\right]$. To evaluate this requires the joint distribution of the all the ε_i's and $\widetilde{R}^m$ together. For now, I assume they are independent so we can evaluate the expectation of the product as the product of expectations:

$$\begin{aligned}\mathbb{E}\left[v\left(\widehat{\widehat{R^e}}(\mathcal{L})\,E\right)\right] &= \mathbb{E}\left[-\prod_{i\in\mathcal{L}}\left|v\left(\widehat{\widehat{R^l}}_i l_i + R^d d_i - \widetilde{R}^m(l_i+d_i)\right)\right|\cdot|v(Z)|\right]\\ &= -\prod_{i\in\mathcal{L}}\mathbb{E}\left[\left|v\left(\widehat{\widehat{R^l}}_i l_i + R^d d_i - \widetilde{R}^m(l_i+d_i)\right)\right|\right]\cdot\mathbb{E}\left[|v(Z)|\right]\end{aligned}$$

Thus, once it can be evaluated, we have $\mathbb{E}\left[v\left(\widehat{\widehat{R^l}}l + R^d d - \widetilde{R}^m(l+d)\right)\right]$ as the appropriate index function that represents the banks preference ranking over the set of loan applications.

Or put formally: let $\succsim$ indicate the weak preference relation represented by the value function of the bank, then:

$$\mathtt{l}_i \succsim \mathtt{l}_j \iff \mathbb{E}\left[v\left(\mathtt{l}_i\,|\mathtt{l}_i \text{ approved}\right)\right] \geq \mathbb{E}\left[v\left(\mathtt{l}_j\,|\mathtt{l}_i \text{ approved}\right)\right]$$

Distribution of per dollar gross return on money market I do not wish to model this features as a joint problem with all the others in this model at present, so I make the stark assumption that all banks at all times believe that $\widetilde{R}^m$ is distributed independently and uniformly over the interval $\left[R^{\underline{r}}, R^{\overline{r}}\right]$

Anticipated distribution of the gross return on an arbitrary loan Consider an arbitrary loan application characteristized by $\left[\begin{array}{cccc} g, & l, & \widehat{\widehat{\alpha}}, & d\end{array}\right]$ that is currently being reviewed.

From the point of view of the bank, this is a stochastic investment opportunity with the following characterization:

The anticipated realized gross return on a loan is a piecewise defined function:

- if the loan applicant is lucky enough (i.e. has a large enough positive idiosyncratic error draw) she will be able to pay back the entire loan plus contracted interest, $R^l(g)\,l$ (call this a "good loan"), from the return on investment of $a = \left(\frac{1+g}{g}\right)l$ in her private asset;
- if paying the full amount of the loan would leave her less than $(1-\delta)$ of her gross earnings, she is declared insolvent and has to repay only a fixed fraction δ of her gross earnings, which the bank believes to be governed by the process $\widehat{\widehat{\alpha}}e^{\varepsilon}$. The bank thus anticipates that it will receive $\delta P\widehat{\widehat{\alpha}}e^{\varepsilon}\left(\frac{1+g}{g}\right)l$ if ε is such that the borrower is insolvent (call this a "bad loan").

Note that both scenarios yield gross returns as a linear function[14] of the size of the loan - this greatly simplifies the nature of the characterization of and solution for optima.

Summarizing formally: The nominal gross return that the bank believes it will receive on each dollar of l, denoted $\widehat{\widehat{R^l}}$, is given by the piecewise defined function:

$$\widehat{\widehat{R^l}} = \begin{cases} \delta\left(P\widehat{\widehat{\alpha}}e^{\varepsilon}\right)\left(\frac{1+g}{g}\right) & \text{if } \delta\left(P\widehat{\widehat{\alpha}}e^{\varepsilon}\right)\left(\frac{1+g}{g}\right) < R^l(g) \\ R^l(g) & \text{otherwise} \end{cases}$$

Let the bank's beliefs be that the loan will turn out to be good with probability $\widehat{\widehat{\pi}}$, then:

$$\begin{aligned} 1-\widehat{\widehat{\pi}} &= \Pr(\text{loan turns out to be bad}) \\ &= \Pr\left(\delta\left(P\widehat{\widehat{\alpha}}e^{\varepsilon}\right)\left(\frac{1+g}{g}\right) < R^l(g)\right) \\ &= \Pr\left(\varepsilon < \ln\left(\frac{R^l(g)}{\delta P\widehat{\widehat{\alpha}}\left(\frac{1+g}{g}\right)}\right)\right) \\ &= F\left(\ln\left(\frac{R^l(g)}{\delta P\widehat{\widehat{\alpha}}\left(\frac{1+g}{g}\right)}\right)\right) \end{aligned}$$

where F is the probability distribution function that governs ε, with associated density function f, assumed to be common knowledge.

Let $\widehat{\widehat{\underline{\varepsilon}}} = \ln\left(\frac{R^l(g)}{\delta P\widehat{\widehat{\alpha}}\left(\frac{1+g}{g}\right)}\right)$.

We can thus summarize: the typical bank's beliefs regarding the process that governs gross returns from investing in an arbirary loan is:

$$\widehat{\widehat{R^l}} = \begin{cases} \delta\left(P\widehat{\widehat{\alpha}}e^{\varepsilon}\right)\left(\frac{1+g}{g}\right) & \text{with probability density } f(\varepsilon)\,F\left(\widehat{\widehat{\underline{\varepsilon}}}\right) \\ R^l(g) & \text{with probability mass } 1-F\left(\widehat{\widehat{\underline{\varepsilon}}}\right) \end{cases}$$

Note that the true returns process of the same loan is identical, except that the true productivity parameter α governs the process rather than the productivity parameter the banks believes to be governing the process.

I collect all of the results and assumptions of this section in the following proposition:

Proposition: Conditional on all the assumptions stated above, the optimal loan approval rule for any bank described in this model is given by Algorithm 1.

[14]This "linearity result" is a direct result of I defined the gearing of a private lender. It need not hold for other definitions

Algorithm 1 Optimal Loan Approval Rule

Step 1: For each loan application evaluate the expected marginal value impact conditional on approval. This results in the vector of real numbers that represent expected loan values $\mathbf{v}{:=}\left\{\mathbb{E}\left[v\left(\mathtt{l}_i\right)\right]\forall i\in\widetilde{\mathcal{L}}\right\}$

Step 2: Sort $\mathbf{v}$ from large to small

Step 3: Starting from the top loan, approve the loans in sequence as long as each of the following holds:

$$\mathbb{E}\left[v\left(\mathtt{l}_i\right)\right]>0$$

$$\sum_{i\in\mathcal{L}} l_i<\overline{L}$$

$$\mathcal{L}\subsetneq\widetilde{\mathcal{L}}$$

2.6.3 The optimal choice of future deposit and loan rate schedules

The last function of the current owners of any bank is to set the rates at which the next periods deposits and loans will be priced.

I assume that owners act as if they will hold exactly the same equity in the same bank next period (nothing prevents this, although it would be more appropriate to assume that owners act to maximize their anticipated reward conditional on simultaneously planning a possible different optimal level of equity holdings, but this is too complicated for now)

That is, $R^d_{t+1}=a$ and $R^l_{t+1}\left(g\right)=b+cg$ are chosen to solve:

$$\max_{a,b,c}\mathbb{E}\left[v\left(\widehat{\widehat{R^e}}_{t+1}\left(\mathcal{L}_{t+1}\right)E_{t+1}\right)|\mathcal{E}_{k,t+1}=\mathcal{E}_{k,t}\right]$$

:

2.7 The Interbank Market

2.7.1 Interbank position in the aggregate

To allow for a full, functional interbank market, I must allow (initially) that any bank can (subject to accounting identities) take any position at any other bank. By definition a bank's position relative to itself is always zero.

To fix notation, suppose that bank k takes position $m_{h,k}$ with promised return $R^m_{h,k}$ at bank h, and holds position $m_{0,k}$ with promised return $R^{\overline{r}}$ at the central bank

The global set of money market and central bank reserve positions at a point in time can be collected as:

$$\underset{[K+1\times K]}{\mathcal{M}} = \begin{bmatrix} m_{0,1} & m_{0,2} & m_{0,3} & \cdots & m_{0,K} \\ 0 & m_{1,2} & m_{1,3} & \cdots & m_{1,K} \\ m_{2,1} & 0 & m_{2,3} & \cdots & m_{2,K} \\ m_{3,1} & m_{3,2} & 0 & & \vdots \\ \vdots & \vdots & & \ddots & m_{K-1,K} \\ m_{K,1} & m_{K,2} & \cdots & m_{K,K-1} & 0 \end{bmatrix}$$

For the development below it is also important that I choose a "accounting convention". for this purpose $m_{h,k}$ is the entry in the balance sheet of bank k. The entry in the balancesheet of bank h is $m_{k,h} = -m_{h,k}$. This means that for computational purposes, I only have to keep track of the $[K \times K]$ upper triangular part of $\mathcal{M}$.

Similarly, $m_{0,h}$ is the position with the central bank in the books of bank h. A central bank reserve requirement $\underline{m}_0 > 0$ could be imposed by requiring: $m_{0,h} \geq \underline{m}_0 \forall h$.

Similary, we can collect all the promised rates of return:

$$\underset{[K+1\times K]}{\mathcal{R}^m} = \begin{bmatrix} R^r_{0,1} & R^r_{0,2} & R^r_{0,3} & \cdots & R^r_{0,K} \\ 1 & R^m_{1,2} & R^m_{1,3} & \cdots & R^m_{1,K} \\ R^m_{2,1} & 1 & R^m_{2,3} & \cdots & R^m_{2,K} \\ R^m_{3,1} & R^m_{3,2} & 1 & & \vdots \\ \vdots & \vdots & & \ddots & R^m_{K-1,K} \\ R^m_{K,1} & R^m_{K,2} & \cdots & R^m_{K,K-1} & 1 \end{bmatrix}$$

My assumptions will lead, in any specific period, to the result that $R^m_{h,k} = R^m_{k,h}$

2.7.2 Functioning of the interbank market

In this section I set up the random matching and Nash bargaining mechanism that I assume clears the interbank markets. The central bank operations will constitute the residual of the interbank market: any bank that still has funds to invest or Recall that the interbank market opens only after all loan approval rules have been carried out. At this stage bank k either has a known cash surplus $\widetilde{M}_k > 0$ that it wishes to invest in the money market, or a cash deficit $\widetilde{M}_k < 0$ it has to borrow on the money market[15]. I refer to these jointly as "desired money market positions".

Define the following index sets of surplus and deficit banks respectively:

$$\mathcal{M}^+ := \left\{ n \middle| \widetilde{M}_n > 0 \right\}$$

[15]It is possible that some banks may have $\widetilde{M}^k = 0$, but these do not enter the money market and can be ignored.

and

$$\mathcal{M}^- := \left\{ h \,\middle|\, \widetilde{M}_h < 0 \right\}$$

First note that nothing ensures that the aggregate cash surplus matches the aggregate cash deficit. In fact, the central bank is always (in this model) the residual position holder in the money market.

Put differently, if the central bank holds no assets other than cash, the total quantity of central bank issued money M_0 in the economy is given by:

$$M_0 = \sum_{k=1}^{K} m_{0,k} = \sum_{n \in \mathcal{M}^+} \widetilde{M}_n + \sum_{h \in \mathcal{M}^-} \widetilde{M}_h.$$

On the lefthand side is nominal money supply and on the right hand side, nominal money demand.

I discuss first the surplus available from an arbitrary match and how it is shared between the two parties of the match, then how the open positions on the two sides of the the interbank market are matched.

Surplus and surplus sharing: For any randomly matched pair $\{n \in \mathcal{M}^+, h \in \mathcal{M}^-\}$ of banks, the desired market positions, $\left\{\widetilde{M}_n, \widetilde{M}_h\right\}$,are of opposite sign but will, in general, not be of equal magnitude. I assume that the smaller of the two determines the transaction size that takes place:

$$m_{h,n} = \min \left\{ \left|\widetilde{M}_n\right|, \left|\widetilde{M}_h\right| \right\}$$

Since all cash deficit banks can lend indefinitely at $R^{\overline{r}}$ from the central bank and all cash surplus banks can invest at $R^{\underline{r}}$ at the central bank, the maximum surplus available in this typical pairwise transaction is:

$$\left(R^{\overline{r}} - R^{\underline{r}}\right) m_{h,n}$$

It is obvious that the lending bank prefers that the rate of return on this transaction be the highest (par with central bank's lending rate), while the borrowing bank prefers that it be the lowest.

I have yet to explore what would be a "market consistent" rule here, but for purposes of the first rounds of computation I assume without further motivation that a Nash bargain occurs where the agreed promised rate of return on this transaction is a weighted geometric average of the two extreme rates of return, with the relative equity capitalization as weights. I.e. I assume that the outcome of the bargaining is a rate

$$R^m_{h,n,t} = \left(R^{\overline{r}}_t\right)^{\left[\frac{E_{n,t}}{E_{n,t}+E_{h,t}}\right]} \left(R^{\underline{r}}_t\right)^{\left[\frac{E_{h,t}}{E_{n,t}+E_{h,t}}\right]}$$

I assume the remainder of any match is invested at or borrowed from the central bank:

$$\widetilde{M}_n + \widetilde{M}_h = \begin{cases} m_{0,n} & \text{if } \widetilde{M}_n + \widetilde{M}_h \geq 0 \\ m_{0,h} & \text{otherwise} \end{cases}$$

Algorithm 2 Money Market Matching and Surplus Sharing

Step1: In general, the number of elements in and $\mathcal{M}^+$ and $\mathcal{M}^-$ will not be the same. If they differ, add zeros to the smaller set until they have equal number of elements. This corresponds to selecting the central bank as the counterparty for those banks in the larger set that would not otherwise have had a match. The central bank absorbs any residual, so I set it's "money demand" or "supply" relative to a specific bank as zero ($\widetilde{M}_0 = 0$).

Step2: Let $\ddot{\mathcal{M}}^+$ and $\ddot{\mathcal{M}}^-$ be randomly permuted vectors of equal length containing the indices of the banks in the corresponding sets. The list of index pairs is the matches for the period in interbank market is the rows of the two column matrix $\ddot{\mathcal{M}} = \left[\ddot{\mathcal{M}}^+ \ddot{\mathcal{M}}^-\right]$.

Step3: for each entry $[n, h] \in \ddot{\mathcal{M}}$, the following is defined as above:
$m_{n,h} = \min\left\{\left|\widetilde{M}_n\right|, \left|\widetilde{M}_h\right|\right\} = -m_{h,n}$
$R^m_{n,h} = \left(R^{\overline{r}}\right)^{\left[\frac{E_n}{E_n+E_h}\right]} \left(R^{\underline{r}}\right)^{\left[\frac{E_h}{E_n+E_h}\right]}$
$m_{0,n} = \begin{cases} \widetilde{M}_n + \widetilde{M}_h & \text{if } \widetilde{M}_n + \widetilde{M}_h \geq 0 \\ 0 & \text{otherwise} \end{cases}$
$R^r_{0,n} = R^{\underline{r}}$ - in words: if a commercial bank has a cash surplus after its match, it receives the "deposit" repo rate at the central bank.
$m_{0,h} = \begin{cases} 0 & \text{if } \widetilde{M}_n + \widetilde{M}_h \geq 0 \\ \widetilde{M}_n + \widetilde{M}_h & \text{otherwise} \end{cases}$
$R^r_{0,h} = R^{\overline{r}}$ - in words: if a commercial bank has a cash deficit after its match, it must pay the "lending" repo rate at the central bank.

Random Matching: For computational speed I prefer an simple once-off match algorithm that can be thought of to "imbed" a type of temporal friction in the market.

To be precise, I assume that each bank gets only one money market match every period according to the following algorithm:

In sum - commercial banks are randomly matched with either another commercial bank, in which case the interbank surplus is shared and the bank with a residual position takes it up at the central bank. If the commercial bank is matched with the central bank, it must take its full position at the central bank at the corresponding rate of return.

2.8 Central Bank Operations

2.8.1 Central Bank Resource Constraint

I base the discussion of the central bank on Reis [2013] and Hall and Reis [2013].

In my model economy there is no cash (in the sense of bank notes and coins) so the only liability of the central bank is the reserves held by commercial banks at the central bank that pays the repo deposit rate rate and the only asset $R^{\overline{r}}$. Since "cash" is only used once in each period to turn realized returns into consumption and new investment / savings, the velocity of money is identically equal to 1 at all dates.

Recall that commercial bank k in period t holds $m_{0,k,t}$ of central bank reserves or debt, where this is a negative value (i.e. a liability) in the budget constraint of the central bank if the commercial bank is a "surplus" bank. These can be thought of as stochastic money demand functions in the repo interest rate spread: $\frac{R_t^{\overline{r}}}{R_t^{\underline{r}}}$ - the demand for money reduces when the expected cost of financing money deficits increases, which increases in the repo interest spread, this in turn reduces the level of lending in the economy and hence suppresses growth to come extent. However it also has allocative impacts - a higher spread means only on average higher productivity assets will get financed which may increase growth.

The central bank balance sheet (in a classical monetary world where the central bank does not attempt to generate seignorage dividends) is thus given by the budget constraint:

$$\sum_{h\in\mathcal{M}_{t+1}^{-}} m_{0,h,t+1} + \sum_{n\in\mathcal{M}_{t+1}^{+}} m_{0,n,t+1} \quad = \quad R_t^{\overline{r}} \sum_{h\in\mathcal{M}_t^{-}} m_{0,h,t} + R_t^{\underline{r}} \sum_{n\in\mathcal{M}_t^{+}} m_{0,n,t}$$

An interpretation of this equation that is particularly useful to conceptualize "monetary policy in this model is the following: The budget constraint of the central bank is a promise to pay gross interest $R_t^{\underline{r}}$ on any reserves kept at the bank. It can keep this promise, as it pays the that it must issue enough money in period $t+1$ to pay the the fraction of promised return on reserves kept at the bank that is not

If I normalize $R_t^{\underline{r}} = 1 \forall t$ (i.e. zero net return), the relationship between money and the repo rate is fixed in this economy:

$$R_t^{\overline{r}} = \frac{\left[\sum_{h\in\mathcal{M}_{t+1}^{-}} m_{0,h,t+1} + \sum_{n\in\mathcal{M}_{t+1}^{+}} m_{0,n,t+1}\right] - \sum_{n\in\mathcal{M}_t^{+}} m_{0,n,t}}{\sum_{h\in\mathcal{M}_t^{-}} m_{0,h,t}}$$

This conception maps neatly into a situation where the central bank attempts to set an interest rate and allows the market to determine the level of money required for this interest to hold - since no bank individually can predict its money demand, neither can the central bank.

Note that the central bank only manages money in this economy and does not enter any other market. If we allow the central bank to buy other assets, e.g. to allow operations such as the successive rounds of "quantitative easing" by purchasing toxic or other assets from banks in trouble, we will have to bring that into its resource constraint/balance sheet.

3 Aggregation and Macroeconomic Accounting Identities

3.1 Real GDP, Aggregate Investment and Consumption

Denote aggregate real consumption by $C_t = \sum_i c_{i,t}$ and aggregate real productive investment $A_t = \sum_i a_{i,t}$. This means real GDP from the demand side is $Y_t = (C_t + A_t)$.

From the supply side, the total of available aggregate real resources Y_t is the aggregate consequence of individual investment decisions in the period $t-1$:

$$Y_t = \frac{1}{P_{t-1}} \sum_{i=1}^{N} w_{i,t}$$

where

$$P_{t-1} w_{i,t} = \begin{cases} R^e_{t-1} e_{i,t-1} + R^d_{t-1} d_{i,t-1} & \text{if } a_{i,t-1} = 0 \\ R^a_{i,t-1} a_{i,t-1} - \min\left\{\delta R^a_{i,t-1} a_{i,t-1}, R^l_{i,t-1}\left(g_{i,t-1}\right) l_{i,t-1}\right\} & \text{if } a_{i,t-1}, l_{i,t-1} > 0 \\ R^a_{t-1} a_{i,t-1} + R^d_{t-1} d_{i,t-1} & \text{if } a_{i,t-1}, d_{i,t-1} > 0 \end{cases}$$

Since the financial claims above must cancel out in the aggregate (that is how $R^e$for instance is determined), we can also state this as:

$$(C_t + A_t) = Y_t = \sum_{i \in \mathcal{A}_{t-1}} \frac{R^a_{i,t-1}}{P_{t-1}} a_{i,t-1}$$

3.2 Nominal Aggregates and Price determination

In this model, there is no individual firm level pricing decision, but since the ultimate size of the money stock is never predictable, real and nominal GDP will not move in tandem. That is, I assume that aggregate prices that prevail in this economy are deterministic outcome of the level of real production and the size of the monetary base.

Since "cash" is only used once in each period to turn realized returns into consumption and new investment / savings, the velocity of money is identically equal to 1 at all dates. This means that the nominal identity that governs price determination in this model is:

$$\frac{\sum_{h \in \mathcal{M}_t^-} m_{0,h,t} + \sum_{n \in \mathcal{M}_t^+} m_{0,n,t}}{P_t} = Y_t = \left(\sum_{i \in \mathcal{A}_{t-1}} \frac{R^a_{i,t-1}}{P_{t-1}} a_{i,t-1} \right)$$

Denote gross growth in real GDP Γ_t and net growth γ_t
so that

$$\begin{aligned}
R_t^{\overline{r}} &= \frac{M_{t+1}^0}{M_t^0} \\
&= \frac{P_{t+1}}{P_t} \frac{\left(\sum_{i\in\mathcal{A}_t} \frac{R_{i,t}^a}{P_t} a_{i,t}\right)}{\left(\sum_{i\in\mathcal{A}_{t-1}} \frac{R_{i,t-1}^a}{P_{t-1}} a_{i,t-1}\right)} \\
&= \Pi_{t+1}\Gamma_{t+1} \\
R_t^{\overline{r}} &= \pi_{t+1} + \gamma_{t+1} + \pi_{t+1}\gamma_{t+1}
\end{aligned}$$

This is just the Fischer relation for this model: nominal interest is the real rate (which must in equilibrium be equal to the growth rate) of the economy plus inflation. It is somewhat different in the time dimension in that the current nominal rate is will in equilibrium be equal to the expected inflation plus the *expected growth.*

What is crucial to note is that growth in any period is a function of the set of assets that are invested in which in turn depends on the loan decision of banks which is influenced via their cost of funds by $R^{\overline{r}}$ (or if the return on cash is not normalized, on the spread of the return/cost of money..

The final consequence of a specific *choice* of$R^{\overline{r}}$ will take some time to work through this economy, as it affects not only inflation conditional on any expected growth rate, but since it also affects the expected financing cost of any loan, it affects which assets are invested in and so real economic outcomes. As such adequately characterizing optimal monetary policy is not yet obvious in this economy - i will be working on that next.

For initial simulations, we can use a simple Taylor rule to close the model:

$$R_t^{\overline{r}*} = \Pi_t^{\phi_\pi}\Gamma_t^{\phi_y}\mathsf{e}^{\nu_t}$$

4 Solution Method

Claim: Starting from an initial condition for the macro economy:

$$\begin{aligned}
P_0 &= 1 \\
Y_0 &= \sum_i w_{i0} \\
\widehat{\boldsymbol{\alpha}}_{j,0} &= \begin{bmatrix} \alpha_1\mu_{1,j,0} \\ \vdots \\ \alpha_i\mu_{i,j,0} \\ \vdots \\ \alpha_N\mu_{N,j,0} \end{bmatrix} \\
M_{0,0} &= Y_0 \\
R_1^r \quad &and \quad R_1^{\overline{r}}
\end{aligned}$$

all the optimizing decisions above can be solved sequentially and an a time path of the economy simulated.

4.1 Envisioned Outcomes

for any sequence of observation perterbations μ and stochastic error draws ε, I can construct:

The full time path of the income, consumption and investment distributions of the economy that functions according to the mechanisms constructed above.

One could then compare the outcomes for several sequences to build look for regularities in the evolotion of these distributions in response to changes to exogenous regulatory and/or policy environment.

For instance: maintaining higher interest rates on average would make some investments in private technology unprofitable which will tend to lower welfare. On the other hand, it will automatically mean that only the more productive technologies are invested in, and moreover, at higher gearing (i.e. there will be more resources available for investment in more productive technologies) which would tend to increase welfare. Which dominates when is the type of question I am interested in.

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Appendices

A.1 Analysis of the Return distribution on an arbitrary loan

the banks beliefs regarding an arbirary loan is:

$$\widehat{\widehat{R^l}} = \begin{cases} R^l\left(g\right) & \text{with probability mass } \pi^* \\ \delta\left(\widehat{\widehat{\alpha}} e^{\varepsilon}\right)\left(\frac{1+g}{g}\right) & \text{with probability density} \phi\left(\varepsilon\right)\left[1-\pi^*\right] \end{cases}$$

marginal impacts of g

$$\frac{\partial \widehat{\widehat{R^l}}}{\partial g} = \begin{cases} R^{l\prime}(g) & > 0 \text{ (by assumption)} \\ -\delta\left(\widehat{\widehat{\alpha}} e^{\varepsilon}\right)\left(\frac{1}{g^2}\right) & < 0 \end{cases}$$

$$\frac{\partial \widehat{\widehat{\pi}}}{\partial g} = -\left(\frac{R^{l\prime}(g)\,\delta\widehat{\widehat{\alpha}}\left(\frac{1+g}{g}\right) + \delta\left(\widehat{\widehat{\alpha}}\right)\left(\frac{1}{g^2}\right)R^l(g)}{\left[\delta\widehat{\widehat{\alpha}}\left(\frac{1+g}{g}\right)\right]^2}\right)\frac{\delta\widehat{\widehat{\alpha}}\left(\frac{1+g}{g}\right)}{R^l(g)} f\left[\ln\left(\frac{R^l(g)}{\delta\widehat{\widehat{\alpha}}\left(\frac{1+g}{g}\right)}\right)\right] < 0$$

Higher gearing thus has one positive and two negative effects: the return per dollar if the borrower is successfull is higher, but the likelihood of a success is lower and the outcome under failuer is worse. So it is not immediate whether a bank prefers higher or low gearing - it will depend crucially on parameters that describe the bank's preferences and the macro conditions and the other loans available

marginal impact of $\widehat{\widehat{\alpha}}$

$$\frac{\partial \widehat{\widehat{R^l}}}{\partial \widehat{\widehat{\alpha}}} = \begin{cases} 0 & \\ \delta\left(e^{\varepsilon}\right)\left(\frac{1+g}{g}\right) & > 0 \end{cases}$$

$$\frac{\partial \widehat{\widehat{\pi}}}{\partial \widehat{\widehat{\alpha}}} = \frac{1}{\widehat{\widehat{\alpha}}} f\left[\ln\left(\frac{R^l(g)}{\delta\widehat{\widehat{\alpha}}\left(\frac{1+g}{g}\right)}\right)\right] > 0$$

As expected, this impact is weakly positive so a bank always prefers high expected return projects.